Moving Between Euler and Lagrangian

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1 Introduction

In these notes we outline how the compressible Euler equations change when the reference frame changes from Eulerian to Lagrangian.

2 Change of Reference Frame

For simplicity, we first start with the conservation of mass in 1D in Eulerian coordinates,

$$\frac{\partial \rho}{\partial t}(x,t) + \frac{\partial(\rho v)}{\partial x}(x,t) = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

where ρ and v are density and velocity, respectively. Applying chain rule, we have,

$$\frac{\partial \rho}{\partial t}(x,t) + v\frac{\partial \rho}{\partial x}(x,t) + \rho\frac{\partial v}{\partial x}(x,t) = 0. \tag{2}$$

Tranform the density to the specific volume; $\rho \mapsto \tau(\rho)$, by $\tau = \rho^{-1}$. Then we have,

$$-\frac{1}{\tau^2}\frac{\partial \tau}{\partial t}(x,t) - \frac{v}{\tau^2}\frac{\partial \tau}{\partial x}(x,t) + \frac{1}{\tau}\frac{\partial v}{\partial x}(x,t) = 0. \tag{3}$$

Multiplying the previous equation by $-\tau$, we have the following,

$$\rho(x,t)\frac{\partial \tau}{\partial t}(x,t) + \rho(x,t)v(x,t)\frac{\partial \tau}{\partial x}(x,t) - \frac{\partial v}{\partial x}(x,t) = 0. \tag{4}$$

Next let $\Phi(\cdot,t) : \mathbb{R} \to \mathbb{R}$ denote the mapping of the fluid particle defined by, $v(\Phi(\xi,t),t) = \frac{\partial \Phi}{\partial t}(\xi,t)$ and $x = \Phi(\xi,t)$. Applying the chain rule with this mapping, note the following identity,

$$\frac{\partial}{\partial t}(\tau(\Phi(\xi,t),t)) = \frac{\partial \tau}{\partial t}(\Phi(\xi,t),t) + \frac{\partial \tau}{\partial x}(\Phi(\xi,t),t)\frac{\partial \Phi}{\partial t}(\xi,t). \tag{5}$$

Using this identity, the conservation of mass in the Lagrangian frame is,

$$\rho(x,t)\frac{\partial}{\partial t}(\tau(\Phi(\xi,t),t)) - \frac{\partial v}{\partial x}(\Phi(\xi,t),t) = 0. \tag{6}$$