Conservation Laws Notes

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1 Preface

I have created these notes mostly as an exercise to strengthen my own knowledge on the subject. I can not guarantee that everything in these notes in 100% correct, as they have note yet been reviewed. However I have attempted to keep things as accurate to the subject as possible. If you find any typos, incorrect statements, or inconsistencies then please let me know by email about this. Additionally, I have written these notes with advanced undergraduate students as the target audience.

2 1D Scalar Conservation Laws

2.1 Introduction

We begin with the general form for a scalar 1D conservation law. That is, we wish to find $u := u(x,t)$ that satisfies,

$$\begin{cases} u_t + f(u)_x = 0, \\ u(x,0) = g(x), \end{cases} \tag{1}$$

where $x \in \mathbb{R}$, $t > 0$, $u : \mathbb{R} \times (0,\infty) \to \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$. The subscripts t and x denote partial derivatives and to clarify the notation further, we can write $f(u)_x = (f(u))_x$. Also, f is commonly referred to as the *flux* function.

Throughout this chapter, we will focus on solving this PDE for a several different types of flux functions. To those more versed in the subject, we avoid discussion of function spaces until the end of the chapter and simply approach the problems in a more elementary mindset.

2.2 The Transport Equation

The simplest nontrivial conservation law is the transport equation, which is,

$$\begin{cases} u_t + a u_x = 0, \\ u(x,0) = g(x). \end{cases} \tag{2}$$

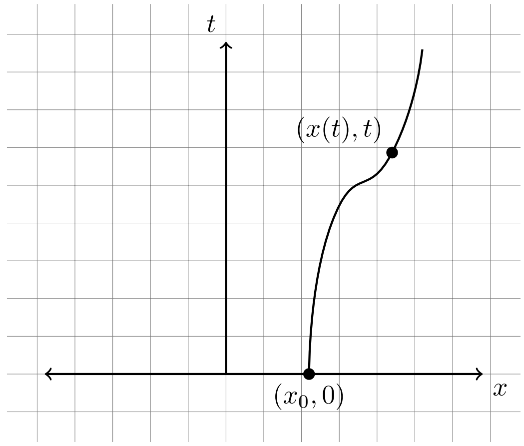

Figure 1: An arbitrary parametrized curve.

Note that our flux function is simply $f(u) = au$. The idea for solving this PDE, is to restrict ourselves to a 1-dimensional curve residing in our domain, $\mathbb{R} \times (0, \infty)$. We do this by first **parametrizing** a curve as, $t \mapsto (x(t), t)$ and set $x(0) = x_0$. So our spatial coordinate is parametrized by the temporal parameter t; see Figure 1. The technique we are employing is called *the method of characteristics*. The trick now, is to differentiate u with respect to t and compare with the PDE. So applying the multivariate chain rule, we have,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}. \tag{3}$$

Now, if $dx/dt = a$, then from (1) we have that $du/dt = 0$. This gives us two ordinary differential equations; lets write them out more clearly,

$$\begin{cases} \dfrac{dx}{dt} = a, \\ x(0) = x_0, \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{du}{dt} = 0, \\ u(x(0), 0) = u(x_0, 0) = g(x_0). \end{cases} \tag{4}$$

The crucial idea here, is that our solution u is **constant** on the curve $(x(t), t)$. Its value does not change for all time, t. But, if we change to a different curve (by changing x_0), then u will take on, a possibly different value. If this is not quite clear, then wait until the end to see a more informative figure. In any case, we have two differential equations that we can solve.

Our solutions to the ODEs are $x(t) = at + x_0$ and $u(x(t), t) = g(x_0)$. Thus our *characteristic* curve is in fact just a line. In order to extend our solution to the whole domain, it is simply a matter of noticing that our characteristic lines fill the half-plane. Simply put, we solve for x_0. So, $x_0 = x - at$, and our solution is then,

$$u(x, t) = g(x - at). \tag{5}$$

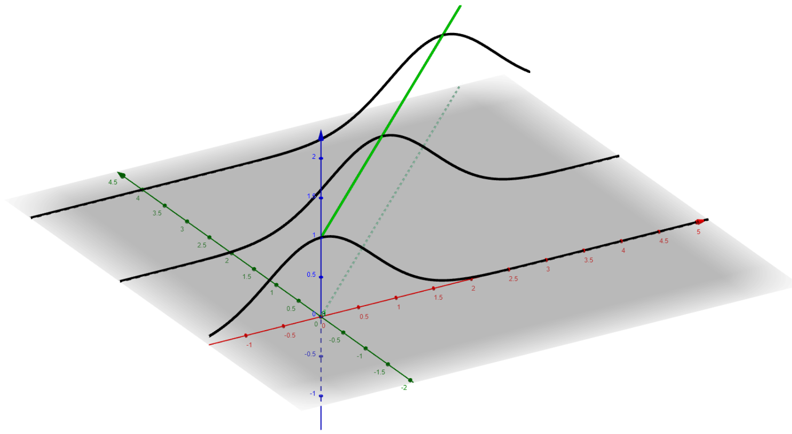

Figure 2: The red, green and blue axes represent the x, t, and solution u, respectively. The green dotted line is the characteristic curve for $x_0 = 0$ and the solid green line is simply displaying the constant value of the function along the characteristic.

This solution is simply the translation of the initial data g to the right with speed a, hence, why we call (2), the *transport equation*. Now that we've found the solution, the idea of the characteristic lines might make more sense by observing Figure 2.

As a final remark, this is not the only way to find the solution to the transport equation. Other methods can be found in Haberman(?) and Evans(?).

Exercise 1. Check that $u(x,t) = g(x - at)$ is in fact a solution to transport equation (2).

Exercise 2. Employ the method of characteristics to solve the transport equation with a constant source. That is, solve the PDE,

$$\begin{cases} u_t + au_x = b, \quad b > 0, \\ u(x,0) = g(x). \end{cases} \tag{6}$$

2.3 The Burgers' Equation (Part 1)

We now move on to our first nonlinear conservation law, which is the Burgers' equation. The Burgers' equation is (1) with the flux function, $f(u) = \frac{1}{2}u^2$. Now, we will assume that our solution u is C^1. So then we can apply the chain rule and you will find that (1) becomes,

$$\begin{cases} u_t + uu_x = 0, \\ u(x,0) = g(x). \end{cases} \tag{7}$$

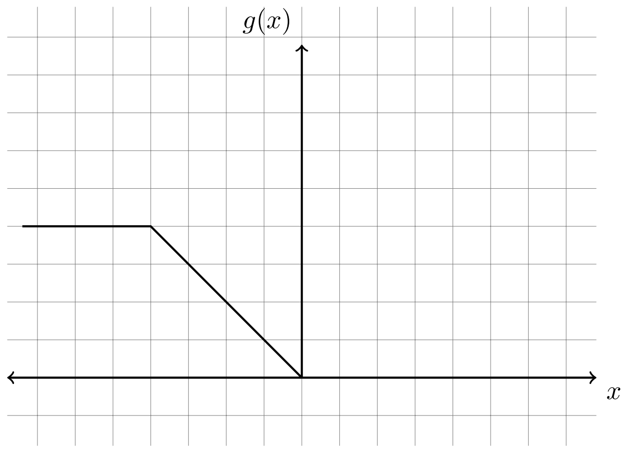

Figure 3: Caption

Remark 1. We cannot always apply chain rule in this way. It is only possible to do this if our solution u is differentiable. We will see by the end of this section and the next, why this is an issue.

(In the next section, make note that applying chain rule in this way is not always possible as our solution must be differentiable.) Specifically, this conservation law is *quasi*-linear; that is, the equation is linear with respect to the highest order derivatives. In order to demonstrate the nature of this PDE, we will look at a specific example. So consider the case when the initial data is,

$$g(x) = \begin{cases} 1, & x < -1, \\ -x, & -1 \leq x \leq 0, \\ 0, & x > 0. \end{cases} \tag{8}$$

A graph of the function is given in Figure 3. Now, we still solve the Burgers' equation the same way we did with the transport equation. Although, this time, our differential equations become,

$$\begin{cases} \dfrac{dx}{dt} = u(x(t), t), \\ x(0) = x_0, \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{du}{dt} = 0, \\ u(x(0), 0) = u(x_0, 0) = g(x_0). \end{cases} \tag{9}$$

Solving these differential equations, we again find that u is constant on the characteristic curves $(x(t), t)$ and the characteristic curves are actually lines given by,

$$x(t) = u(x(t), t)t + x_0 = g(x_0)t + x_0. \tag{10}$$

What is important to see, is that our characteristic lines are dependent on our initial data $g(x)$. We have plotted the characteristic lines in Figure 4 and make note of the time t_b when the lines intersect. The fact that these lines

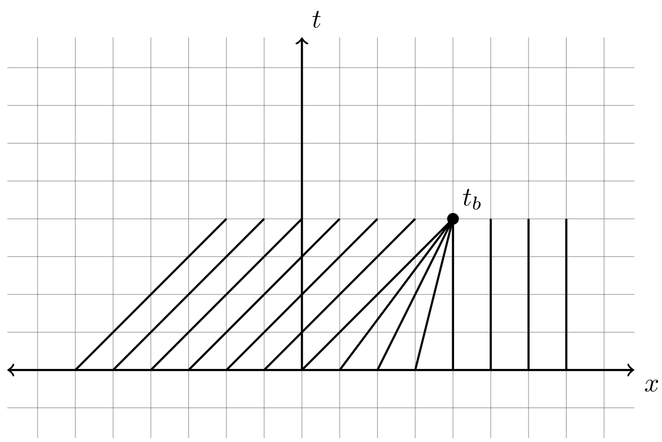

Figure 4: Note that the slopes of the lines are determined by the initial data, $g(x)$. Inspect (8) to see that the picture makes sense. (Don't forget we are plotting x vs. t.)

intersect is a problem. Before reading onward, think about why this would be an issue.

Recall that our solution is constant on each characteristic, but it *doesn't* imply that our solution is the same on each characteristic. So when $t = t_b$, our function becomes multivalued at $x = 1$. Lets first find t_b, which is often referred to as, the **break time** and define it specifically.

Definition 1. The break time, t_b, is the smallest time for which the characteristics intersect.

Theorem 1. *If u solves the Burgers' equation* (7)*, the initial data is C^1; that is, $g \in C^1(-\infty, \infty)$, and there exists an interval on which g is decreasing. Then a break time t_b exists, and is given by,*

$$t_b = \frac{-1}{\min_{x \in \mathbb{R}} g'(x)}. \tag{11}$$

Proof. We start by simply equating to arbitrary characteristics and solving for t. Let $x_0 < x_1$ be the starting values for two characteristic equations, then we have $g(x_0)t + x_0 = g(x_1)t + x_1$. Solving for t, gives us,

$$t = \frac{x_1 - x_0}{g(x_0) - g(x_1)}. \tag{12}$$

We further rewrite this expression as,

$$t = \frac{-1}{\frac{g(x_1) - g(x_0)}{x_1 - x_0}}. \tag{13}$$

By the mean value theorem, there exists $\xi \in (x_0, x_1)$ such that $g'(\xi) = (g(x_1) - g(x_0))/(x_1 - x_0)$. So $t = -1/g'(\xi)$. But x_0 and x_1 are arbitrary, and we only consider values for which $t \geq 0$. From the hypothesis, there must be an interval for which $g' < 0$, which guarantees that $t \geq 0$. Since the break time is the smallest nonnegative number for which the characteristics intersect, we then take the minimum of g', hence the result. □

Remark 2. We typically only consider initial data which is bounded.

Remark 3. If g is always increasing, then there will not be a break time.

Remark 4. Note that there is a more general theorem for an arbitrary flux, f. We omit this for the time being.

You may have noticed, that the starting example is **not** a C^1 function. However, this will not be an issue since we can talk about derivatives in the weak sense (this is a more advanced topic), when applying the formula. We will not get into weak derivatives, but for completeness, we present the weak derivative of g as,

$$g'(x) = \begin{cases} -1, & -1 < x < 0, \\ 0, & x < -1 \text{ or } x > 0. \end{cases} \tag{14}$$

Notice that we have not defined the derivative at the points where g' does not exist in the usual sense. It is now easy to see that the break time is $t_b = 1$.

Before we jump into the problem for time $t > t_b$, we can start by finding our solution for $t < t_b$, using the techniques we developed for the transport equation. Recall our solution on a characteristic was, $u(x(t), t) = g(x_0)$, where $x(t) = u(x(t), t)t + x_0$. Solving for x_0 we can write our solution as,

$$u(x, t) = g(x - u(x, t)t). \tag{15}$$

But this is an implicit equation, which is to complicated. In order to find our solution exactly, we break the problem into three cases.

Case 1: If $x_0 < -1$ then $g(x_0) = 1$ and $x(t) = t + x_0$, hence our solution is simply $u(x(t), t) = 1$.

Case 2: If $-1 \leq x_0 \leq 0$ then $g(x_0) = -x_0$. So our characteristic line becomes,

$$\begin{aligned} x(t) &= g(x_0)t + x_0 \\ &= -x_0 t + x_0, \end{aligned}$$

and solving for x_0 we have, $x_0 = x/(1 - t)$. Then our solution becomes,

$$u(x(t), t) = g(x_0) = -x_0 = -\frac{x(t)}{1 - t}.$$

Case 3: If $x_0 > 0$, then $g(x_0) = 0$ and $x(t) = x_0$ and hence $u(x, t) = 0$.

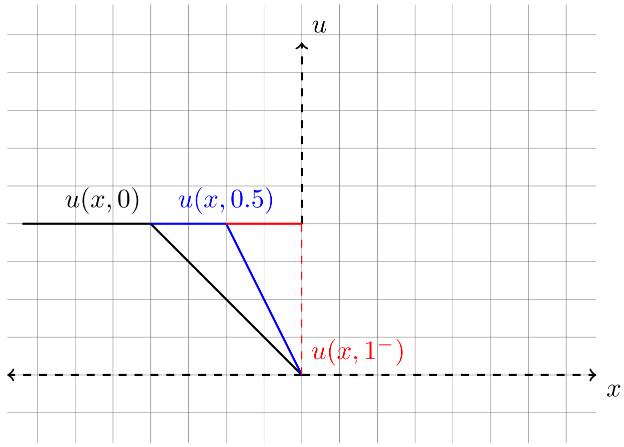

Figure 5: Solution, $u(x,t)$ plotted for three different times: $t = 0$, $t = 0.5$ and $t \to 1^-$. Note that the discontinuity forms at the break time $t_b = 1$.

Therefore, our solution can be written as,

$$u(x,t) = \begin{cases} 1, & x_0 < -1, \\ \frac{-x}{1-t}, & -1 \leq x_0 \leq 0, \\ 0, & x_0 > 0. \end{cases} \tag{16}$$

Lastly, we need to change x_0 to the variables x and t. Using the identities for the characteristic equations in the three prior cases, our solution is thus,

$$u(x,t) = \begin{cases} 1, & x < t - 1, \\ \frac{-x}{1-t}, & t - 1 \leq x \leq 0, \\ 0, & x > 0, \end{cases} \tag{17}$$

for $t \in (0,1)$. The plot of this solution is given for several time steps in Figure 5. The key takeaway from Figure 5 is that a discontinuity forms at the break point $t_b = 1$. Even if you have smooth initial data, your solution can eventually become discontinuous. This is a big problem, since our solution $u(x,t)$ must solve (1), i.e., it must be differentiable. We get around this difficulty by reducing requirements of differentiability through the notion of a *weak* solution.

2.4 Weak Solutions

In order to introduce the concept of a weak solution, we first must introduce a special class of functions, which we refer to as *test* functions.

Definition 2. A function ϕ is called a *test* function if it is infinitely differentiable, bounded, and has compact support. We let $C_c^\infty(D)$ denote the space of all such test functions over some domain D.

Those unfamiliar with compact sets are advised to pick up a text in real analysis, e.g., Royden, Rudin, Folland, etc... However, in $\mathbb{R}$ a compact set is simply a closed and bounded set.

Remark 5. By the Heine-Borel theorem a set in $\mathbb{R}^n$ is closed and bounded if and only if it is compact. This is not true in general for other topological spaces.

Definition 3. The support of a function ϕ is defined as,

$$\text{supp}(\phi) = \overline{\{x \in \mathbb{R} \,:\, \phi(x) \neq 0\}}, \tag{18}$$

where the overbar denotes closure of the set. Additionally, we say a function has compact support if $\text{supp}(\phi)$ is compact.

A common example of a test function is,

$$\phi(x) = \begin{cases} e^{-1/(1-x^2)} & |x| < 1, \\ 0 & \text{otherwise} . \end{cases} \tag{19}$$

Note that $\text{supp}(\phi) = [-1, 1]$ and through diligent use of calculus it can be shown that ϕ is in fact infinitely differentiable. Now we are ready to set up our weak solutions to (1).

Definition 4. A function $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is a weak solution to (1) if

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + f(u)\phi_x \, dx \, dt = -\int_{-\infty}^\infty \phi(x, 0)u(x, 0) \, dx, \tag{20}$$

for all $\phi \in C_c^\infty(\mathbb{R} \times [0, \infty))$.

The first thing you might be thinking is, "why is this definition so complicated?" However, it is not as bad as it looks. We can derive this definition by simply multiplying (1) by a test function ϕ, integrating over $\mathbb{R} \times [0, \infty)$, and then applying integration by parts. There are a few things to note about this definition.

Remark 6. If you are not familiar with the Lebesgue space L^∞, then you can simply think of u as being a bounded function.

Remark 7. The definition does not have any derivatives on the solution u or the flux $f(u)$. Therefore, our weak solutions can be discontinuous. So the issue we ran into when trying to solve the Burgers' equation can soon be rectified with the use of weak solutions.

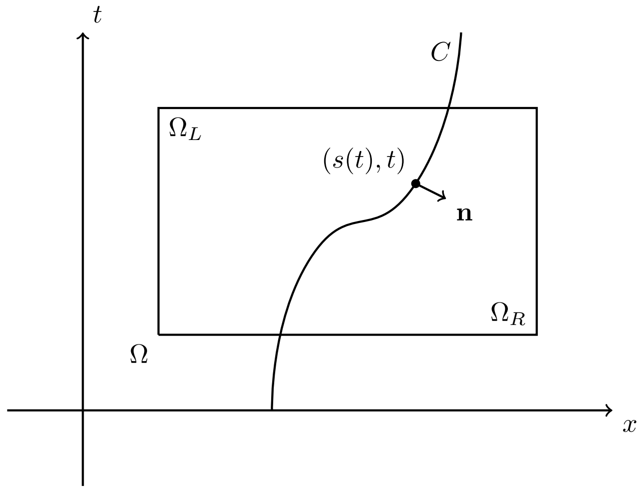

Figure 6: Put something here...

Remark 8. If u is a classical (smooth) solution to (1), then it is also a weak solution to (1). This can be shown by moving the derivatives back onto u and $f(u)$ in (20) by integration by parts and then using an upcoming lemma.

Remark 9. In our definition of a weak solution, the test functions, ϕ, are not necessarily zero at $t = 0$. However, if $\phi \in C_c^\infty(\mathbb{R} \times (0, \infty))$ then $\lim_{t \to 0} \phi(x, t) = 0$.

We are now ready to look at weak solutions which are discontinuous. To do this, we start by deriving the so called, Rankine-Hugoniot condition.

2.4.1 The Rankine-Hugoniot Condition

Let u solve (1) in the weak sense of (20). Then consider the box, $\Omega := [x_0, x_1] \times [t_0, t_1] \subset \mathbb{R} \times (0, \infty)$ of which contains a curve C that bisects Ω into two subdomains. That is, $\Omega = \Omega_L \cup \Omega_R \cup C$. We will parametrize the curve C, by $t \mapsto (s(t), t)$. See Figure 6 for a better visualization. We assume that our solution u, is smooth on both Ω_L and Ω_R separately, but we make no assumptions of continuity on the curve C; i.e., we can think of C as being the curve on which u is discontinuous.

Now let ϕ be a test function which has compact support on Ω. Then by the definition of the weak solution, equation (20) gives us,

$$\int_\Omega u\phi_t + f(u)\phi_x \, dx \, dt = 0.$$

We can separate this integral over the two separate subdomains, Ω_L and Ω_R. Then we wish to apply integration by parts. In order to handle this integration more easily in two dimensions, we can rewrite the integrand in a more useful way. So let, $\nabla_{x,t} = (\partial/\partial x, \partial/\partial t)$, then we have,

$$\int_\Omega u\phi_t + f(u)\phi_x \, dx \, dt = \int_\Omega (f(u), u) \cdot \nabla_{x,t}\phi \, dx \, dt. \tag{21}$$

Remark 10. There are many variations of the integration by parts formula, but in particular, we use the following formula:

$$\int_\Omega \mathbf{u} \cdot \nabla v \, dV = \int_{\partial\Omega} v\mathbf{u} \cdot \mathbf{n} \, dS - \int_\Omega (\nabla \cdot \mathbf{u}) v \, dV, \tag{22}$$

where $\mathbf{u}$ is a vector valued function, v is a scalar function, and $\mathbf{n}$ is the outward pointing normal.

Finally, using the compact support of ϕ, we arrive at the following equations, the integration by parts gives,

$$\begin{aligned}\int_{\Omega_L} (f(u), u) \cdot \nabla_{x,t}\phi \, dx \, dt &= \int_C \phi(f(u_L), u_L) \cdot \mathbf{n}_L \, dS \\ &\quad - \int_{\Omega_L} \nabla_{x,t} \cdot (f(u), u)\phi \, dx \, dt, \\ \int_{\Omega_R} (f(u), u) \cdot \nabla_{x,t}\phi \, dx \, dt &= \int_C \phi(f(u_R), u_R) \cdot \mathbf{n}_R \, dS \\ &\quad - \int_{\Omega_R} \nabla_{x,t} \cdot (f(u), u)\phi \, dx \, dt,\end{aligned}$$

where $\mathbf{n}_L$ and $\mathbf{n}_R$ are the outward point normals on Ω_L and Ω_R respectively. Also, we denote u_L and u_R as the limit of u approaching the curve C from the respective domains, Ω_L and Ω_R.

Remark 11. Note the integration by parts is valid on each subdomain, since we assumed that u was smooth on Ω_L and Ω_R separately.

We now add the two equations together and use the fact that $\mathbf{n}_L = -\mathbf{n}_R$ on C. Therefore, we have the following,

$$\int_C \phi\big[(f(u_R), u_R) \cdot \mathbf{n}_L - (f(u_L), u_L) \cdot \mathbf{n}_L\big] \, dS = 0. \tag{23}$$

Recall the parametrization of C was $t \mapsto (s(t), t)$. So the outward normal is $\mathbf{n}_L = (1, -s'(t))/\sqrt{1 + (s'(t))^2}$. Multiplying everything out, we find that,

$$\int_{t_0}^{t_1} \phi(s(t), t)\big[f(u_R) - f(u_L) - s'(t)(u_R - u_L)\big] \frac{1}{\sqrt{1 + (s'(t))^2}} dt = 0.$$

To conclude the result, we need a famous lemma.

Lemma 1. *(The Fundamental Lemma of Variational Calculus). Let f be a continuous function on (a, b), if*

$$\int_a^b f(x)\phi(x)\,dx = 0 \tag{24}$$

for every $\phi \in C_c^\infty([a, b])$, then $f \equiv 0$.

Because of the smoothness assumptions, the function in the square brackets will be continuous. So since the identity holds for every test function ϕ; and in particular, $\phi/\sqrt{1 + (s'(t))^2}$ is also a test function, we have by the fundamental lemma of variational calculus that,

$$f(u_R) - f(u_L) - s'(t)(u_R - u_L) = 0. \tag{25}$$

This is the famous Rankine-Hugoniot condition, which is also written as

$$[\![f]\!] = s'(t)[\![u]\!]. \tag{26}$$

We call $s'(t)$ the shock speed, and often denote it by S. Several remarks are now in order.

Remark 12. Note that if our solution u is continuous, then the Rankine-Hugoniot condition holds trivially.

Remark 13. In solving problems, we will want to determine the shock speed, $s'(t)$. Since finding this speed will tell us how our discontinuity will propagate. We will see this as we continue our solution of the Burgers' equation.

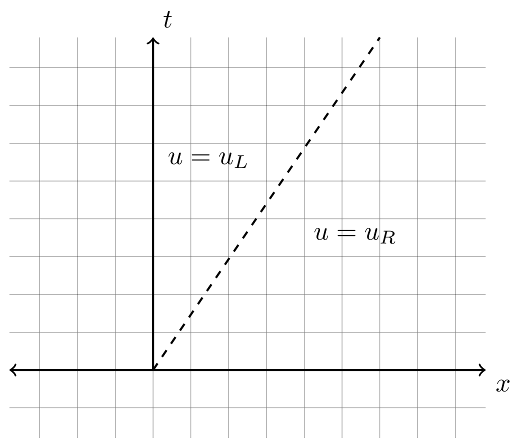

Figure 7: The shock is indicated by a dash line and does not necessarily have to be directed into the first quadrant. If the shock speed is negative, then the shock will be directed into the second quadrant.

2.5 The 1D Riemann Problem

Now that we have the power of the Rankine-Hugoniot condition, we can start finding weak solutions which have discontinuities. Before we get to finishing our Burgers' equation example, we will look at (1) with the simplest nontrivial initial data.

Definition 5. The 1D scalar conservation law (1) with initial data,

$$u(x,0) = \begin{cases} u_L, & \text{if } x < 0, \\ u_R, & \text{if } x > 0, \end{cases} \tag{27}$$

for u_L, u_R constants ($u_L \neq u_R$); is referred to as *the 1D Riemann problem.*

Since we are starting with initial data which is not continuous, the most intuitive approach to solving the Riemann problem will be to invoke the Rankine-Hugoniot condition,

$$f(u_R) - f(u_L) = s'(t)(u_R - u_L). \tag{28}$$

Therefore the shock speed is given by,

$$S := s'(t) = \frac{f(u_R) - f(u_L)}{u_R - u_L}. \tag{29}$$

Recalling the parametrization, $t \mapsto (s(t), t)$, we see that the shock is simply a straight line in the x, t-plane. This tells us how the discontinuity propagates. So we make the simple guess that our (weak) solution is given by,

$$u(x,t) = \begin{cases} u_L, & \text{if } x < St, \\ u_R, & \text{if } x > St. \end{cases} \tag{30}$$

We illustrate the solution in Figure 7. To show this is in fact the weak solution, we have to show that u satisfies equation (20) for every test function ϕ. This process is typically left as an exercise in many books but we will work it out here. So without loss of generality, assume $S > 0$, then we have the following,

$$
\begin{aligned}
L &:= \int_0^\infty \int_{-\infty}^\infty u\phi_t + f(u)\phi_x \, dx \, dt \\
&= \int_{-\infty}^\infty \int_0^\infty u\phi_t \, dt \, dx + \int_0^\infty \int_{-\infty}^\infty f(u)\phi_x \, dx \, dt \\
&= \int_{-\infty}^0 \int_0^\infty u_L\phi_t \, dt \, dx + \int_0^\infty \int_{x/S}^\infty u_L\phi_t \, dt \, dx + \int_0^\infty \int_0^{x/S} u_R\phi_t \, dt \, dx \\
&\qquad + \int_0^\infty \int_{-\infty}^{St} f(u_L)\phi_x \, dx \, dt + \int_0^\infty \int_{St}^\infty f(u_R)\phi_x \, dx \, dt \\
&= -\int_{-\infty}^0 u_L\phi(x,0) \, dx - \int_0^\infty u_L\phi(x, \tfrac{1}{S}x) \, dx + \int_0^\infty u_R(\phi(x, \tfrac{1}{S}x) - \phi(x,0)) \, dx \\
&\qquad + \int_0^\infty f(u_L)\phi(St,t) \, dt - \int_0^\infty f(u_R)\phi(St,t) \\
&= -\int_{-\infty}^\infty u(x,0)\phi(x,0) \, dx + \int_0^\infty \big[f(u_L) - f(u_R) - S(u_L - u_R)\big]\phi(St,t) \, dt.
\end{aligned}
$$

But the last integral contains exactly the Rankine-Hugoniot condition. Hence we arrive at the conclusion. Note that the integrals were split-up over the different domains of integration (it is helpful to draw a picture).

So that's it, right? We found the solution to the Riemann problem and it is always a shock that separates the two states u_L and u_R. We can go home now... Wrong, keep in mind we found **a** weak solution. Who's to say there are not *other* weak solutions.

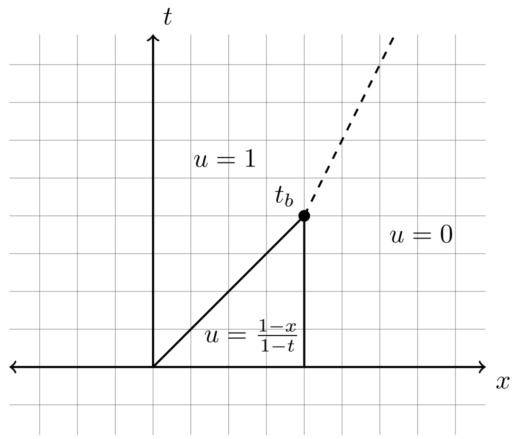

Figure 8: The solution $u(x,t)$ in the x,t-plane. Note that the solution is divided into three separate domains and we use a dashed line to represent the shock (line of discontinuity).

2.6 The Burgers' Equation (Part 2)

You can now argue that

$$u(x,t) = \begin{cases} \begin{cases} 1, & x < t - t, \\ \frac{-x}{1-t}, & t - 1 \leq x \leq 0, \\ 0, & x > 0, \end{cases} & \text{for } t < 1 \\ \begin{cases} 1, & x < \frac{1}{2}t, \\ 0, & x > \frac{1}{2}t. \end{cases} & \text{for } t \geq 1 \end{cases} \tag{31}$$

is the complete weak solution of (7). (We will not work through the weak solution definition for this.)

It can often be informative to look at the solution in the x,t-plane. See Figure 8. Notice in that figure that the solution value of $u = 1$, propagates at a slower rate once the shock is formed. So if we are to draw the characteristic lines for $x < 0$ and $x > 1$, we would see that they will eventually collide with the shock. What we are describing here, is the so called "Lax Entropy Condition" which we will define more rigorously in the next section.