# Applied/Numerical Qualifier Solution: August 2009

Bennett Clayton

Texas A&M University

July 14, 2021

**Problem 1.** Consider the following finite element triple:

- $K =$ a rectangle with vertices $\{a^i\}$, $i = 1, 2, 3, 4$.
- $P = Q^3 = \text{span}\{x_1^i x_2^j\,;\, i, j = 0, \ldots, 3\}$.
- $N = \{p(a^i), p_1(a^i), p_2(a^i), p_{12}(a^i),\ i = 1, 2, 3, 4\}$. (Here $p_i$ denotes differentiation with respect to $x_i$).

**a.** Show that the above finite element is unisolvent.

**Solution**: For simplicity, we will work on the unit square $\hat{K}$ with vertices

$$\{a^i\} = \{(0,0), (1,0), (1,1), (0,1)\}.$$

Working on the reference element is satisfactory due to the affine equivalence of finite elements. Recall the definition:

**Definition**: *Let $\mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^d$, $K \in \mathcal{T}_h$, $P_K$ a finite dimensional vector space of functions defined on $K$, and $\Sigma_K$ the space of linear forms mapping $P_K$ to $\mathbb{R}$. Let further $(\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}})$, be a reference element. Then, the finite elements $(K, P_K, \Sigma_K)$, $K \in \mathcal{T}_h$, are said to be affine equivalent to the reference element $(\hat{K}, \hat{P}_{\hat{K}}, \hat{\Sigma}_{\hat{K}})$, if there exists an invertible affine mapping $F_K : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $K \in \mathcal{T}_h$*

$$K = F_K(\hat{K}) \tag{1}$$

$$P_K = \{p : K \to \mathbb{R} \mid p = \hat{p} \circ F_K^{-1},\ \hat{p} \in \hat{P}_{\hat{K}}\} \tag{2}$$

$$\Sigma_K = \{\ell_i : P_K \to \mathbb{R} \mid \ell_i = \hat{\ell}_i \circ F_K^{-1},\ \hat{\ell}_i \in \hat{\Sigma}_{\hat{K}},\ 1 \le i \le n_K\}. \tag{3}$$

The idea is that the unisolvence on the reference element can be "transferred" to the physical elements. Let $f \in P$, then notice that for $f$ restricted to the edge $[0,1] \times \{x_2 = 0\}$, $f$ is a third degree polynomial. It can be seen that since $f$ is zero at two points and $f'$ is zero at two points, then $f$ will be identically zero on that edge. Similarly, we can show that $f \equiv 0$ on the other three edges of our square. This implies that our function has the following form,

$$f(x_1, x_2) = x_1(x_1 - 1)x_2(x_2 - 1)\gamma(x_1, x_2),$$

where $\gamma$ is a of the form $\gamma(x_1, x_2) = a + bx_1 + cx_2 + dx_1x_2$. Now consider the mixed partial derivative of $f$. Using product rule, we have,

$$\begin{aligned} f_{x_1x_2} = (2x_1{-}1)(2x_2 - 1)\gamma + (2x_1 - 1)(x_2^2 - x_2)\gamma_{x_2} \\ + (x_1^2 - x_1)(2x_2 - 1)\gamma_{x_1} + (x_1^2 - x_1)(x_2^2 - x_2)\gamma_{x_1x_2}. \end{aligned}$$

Note that $(2x_1 - 1)(2x_2 - 1) \neq 0$ for our vertices $\{a^i\}$, but the other three terms will vanish for each $a^i$. Therefore we must have that $\gamma(a^i) = 0$ for $i = 1, 2, 3, 4$. But by definition of $\gamma$, if $\gamma = 0$ for four points then $\gamma \equiv 0$. Thus the finite element is unisolvent. ■

**b.** What do you need to do to check if the above element with a rectangular mesh results in a $C^1$ finite element space?

**Solution:** Let $p, q \in P$, with $p$ defined on $K_1$ and $q$ defined on $K_2$ where $K_1$ and $K_2$ are adjacent elements such that, $p$ and $q$ share the same values of the degrees of freedom on the interface between $K_1$ and $K_2$. Then we must show that the "stiching together" of the functions $p$ and $q$ on the whole domain $K_1 \cup K_2$ results in a $C^1$ function. Specifically, we require that the function,

$$\phi(x, y) := \begin{cases} p(x, y) & \text{if } (x, y) \in K_1, \\ q(x, y) & \text{if } (x, y) \in K_2, \end{cases} \tag{4}$$

be a $C^1$ function. ■

**c.** Does the above element (with a rectangular mesh) result in a $C^1$ finite element space? (Explain your answer).

**Proof:** We follow the plan proposed in part b. Let $a^1 = (x_1, y_1)$ and $a^2 = (x_1, y_2)$ be the points that form the line segment $\ell$; in particular, $\ell$ is a vertical line segment. Then we have,

$$\begin{aligned} p|_\ell(a^1) &= (q|_\ell)(a^1), & p|_\ell(a^2) &= (q|_\ell)(a^2) \\ (p_1)|_\ell(a^1) &= (q_1)|_\ell(a^1), & (p_1)|_\ell(a^2) &= (q_1)|_\ell(a^2) \\ (p_2)|_\ell(a^1) &= (q_2)|_\ell(a^1), & (p_2)|_\ell(a^2) &= (q_2)|_\ell(a^2) \\ (p_{12})|_\ell(a^1) &= (q_{12})|_\ell(a^1), & (p_{12})|_\ell(a^2) &= (q_{12})|_\ell(a^2) \end{aligned}$$

If we define,

$$\begin{aligned} f &= p|_\ell - q|_\ell, \\ g &= p_1|_\ell - q_1|_\ell. \end{aligned}$$

Then $f = f(y)$ is a cubic polynomial. Note that $f(y_2) = f(y_1) = f'(y_2) = f'(y_1) = 0$ which implies that $f$ has 4 roots, hence $f \equiv 0$. By a similar reasoning, we can conclude that $g \equiv 0$. This implies that our function $\phi$ is continuous on $K_1 \cup K_2$. In addition, the partial derivatives exist and are continuous on all of $K_1 \cup K_2$. These results come directly from the shared degrees of freedom. Thus $\phi \in C^1(K_1 \cup K_2)$. A similar argument can be used for a horizontal line segment. This shows that the above element will result in a $C^1$ finite element space. ∎

**Problem 2.** Consider the Neumann Problem:

$$-\Delta u = f \quad \text{in } \Omega \tag{5}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega. \tag{6}$$

Here $\Omega$ is a bounded domain in $\mathbb{R}^2$ and $f$ and $g$ are suitably smooth.

**a.** Derive a weak form of the above problem using a test function in $H^1(\Omega)$.

**Proof:** Multiplying $-\Delta u = f$ by a test function $v \in H^1(\Omega)$ and integrating over $\Omega$, we have,

$$\begin{aligned} -\int_\Omega \Delta u v \, d\boldsymbol{x} &= \int_\Omega \nabla u \cdot \nabla v \, d\boldsymbol{x} - \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds \\ &= \int_\Omega \nabla u \cdot \nabla v \, d\boldsymbol{x} - \int_{\partial\Omega} g v \, ds. \end{aligned}$$

Adding the boundary term to the other side, we have,

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, d\boldsymbol{x} = \int_\Omega f v \, d\boldsymbol{x} + \int_{\partial\Omega} g v \, ds =: L(v).$$

So our weak formulation becomes: find $u \in H^1(\Omega)$ such that $a(u,v) = L(v)$ for all $v \in H^1(\Omega)$.

**b.** Discuss when the weak form of Part a. has a solution and if it is unique.

**Proof:** First note that $a$ and $F$ can both be shown to be continuous (for $F$ you need to invoke the trace lemma). Then from the Lax-Milgram lemma a solution exists and is unique if $a$ is $H^1(\Omega)$-elliptic (coercive). However, in our case $H^1(\Omega)$-ellipticty will not hold since we do not have a Poincarè inequality (or any other means to bound $a(u,u)$ below). Note that $v =$ constant is a permissible function in our test function space. So set $v = 1$ and we have

$$\int_\Omega f \, d\boldsymbol{x} + \int_{\partial\Omega} g \, ds = 0.$$

This is called the *compatibility condition* or *solvability condition* and is necessary for existance of the solution.

Next, note that if $u$ is a solution then $u + c$ is also a solution for $c \in \mathbb{R}$. In order to guarantee a *unique* solution we can impose the condition,

$$\int_\Omega u \, d\boldsymbol{x} = 0.$$

This assumption allows us to derive a Poincarè inequality, which allows us apply the Lax-Milgram lemma. ∎

**c.** Describe a variational formulation of (5) in terms of an appropriate Hilbert space $V$. Be sure to explicitly define $V$.

**Proof:** The variational formulationa will be: find $u \in V := \{v \in H^1(\Omega) \, : \, \int_\Omega v \, d\boldsymbol{x} = 0\}$ such that $a(u,v) = L(v)$ for all $v \in V$. Note we cannot impose both $\partial u/\partial n = 0$ and $u = 0$ (Cauchy boundary condition) on $\partial\Omega$ since then the problem becomes ill-posed.

**d.** Prove coercivity of the form of Part a. on the $V$ of Part c. when $\Omega = (0,1)^2$.

**Proof:** In order to prove coercivity, we a Poincarè type inequality. Specifically, we will prove

$$||u||^2_{L^2(\Omega)} \leq C\Big[\Big(\int_\Omega u\,dx\Big)^2 + ||\nabla u||^2_{L^2(\Omega)}\Big] = C||\nabla u||^2_{L^2(\Omega)},$$

for some constant $C > 0$. We will start by proving the result for a smooth function $\phi \in C^\infty(\Omega) \cap V$. Note we have the following identity,

$$\phi(x_2, y_2) - \phi(x_1, y_1) = \int_{x_1}^{x_2} \frac{\partial \phi}{\partial x}(x, y_2)\,dx + \int_{y_1}^{y_2} \frac{\partial \phi}{\partial y}(x_1, y)\,dy.$$

Squaring both sides, we have,

$$\begin{aligned}\phi^2(x_2, y_2) - 2\phi(x_2, y_2)\phi(x_1, y_1) + \phi^2(x_1, y_1) &= \Big(\int_{x_1}^{x_2} \frac{\partial \phi}{\partial x}(x, y_2)\,dx + \int_{y_1}^{y_2} \frac{\partial \phi}{\partial y}(x_1, y)\,dy\Big)^2 \\ &\leq 2\Big(\int_{x_1}^{x_2} \frac{\partial \phi}{\partial x}(x, y_2)\,dx\Big)^2 + 2\Big(\int_{y_1}^{y_2} \frac{\partial \phi}{\partial y}(x_1, y)\,dy\Big)^2 \\ &\leq 2||\nabla \phi||^2_{L^2(\Omega)}.\end{aligned}$$

Now we integrate both sides of the inequality from 0 to 1 for $x_1$, $x_2$, $y_1$, and $y_2$ (four different variables). Doing so will give us,

$$2||\phi||^2_{L^2(\Omega)} - 2\Big(\int_\Omega \phi\,dx\,dy\Big)^2 \leq 2||\nabla \phi||^2_{L^2(\Omega)}.$$

Thus the result follows where $C = 1$. Now to prove for $u \in H^1(\Omega)$, we use the fact that

$$H^1(\Omega) = \overline{C^\infty(\Omega)}^{||\cdot||_{H^1(\Omega)}}.$$

So for $\varepsilon > 0$, we can find $\phi \in C^\infty(\Omega) \cap V$ such that $||u - \phi||^2_{H^1(\Omega)} < \varepsilon/16$. Now consider,

$$\begin{aligned}||u||^2_{L^2(\Omega)} &\leq ||u||^2_{H^1(\Omega)} \\ &\leq (||u - \phi||_{H^1(\Omega)} + ||\phi||_{H^1(\Omega)})^2 \\ &\leq 2||u - \phi||^2_{H^1(\Omega)} + 2||\phi||^2_{H^1(\Omega)} \\ &\leq \frac{\varepsilon}{8} + 2||\phi||^2_{L^2(\Omega)} + 2|\phi|^2_{H^1(\Omega)} \\ &\leq \frac{\varepsilon}{8} + 4|\phi|^2_{H^1(\Omega)} \\ &\leq \frac{\varepsilon}{8} + 8|u - \phi|^2_{H^1(\Omega)} + 8|u|^2_{H^1(\Omega)}.\end{aligned}$$

Then note that $|u - \phi|^2_{H^1(\Omega)} \leq ||u - \phi||^2_{H^1(\Omega)} < \varepsilon/16$. Thus we have,

$$||u||^2_{L^2(\Omega)} \leq \frac{\varepsilon}{8} + \frac{\varepsilon}{2} + 8|u|^2_{H^1(\Omega)} < \varepsilon + 8|u|^2_{H^1(\Omega)}.$$

Since this holds for any $\varepsilon > 0$, we thus have $||u||_{L^2(\Omega)} \leq C||\nabla u||_{H^1(\Omega)}$. Since we have a Poincarè inequality, the coercivity proof follows in the typical way. ∎

**Problem 3.** Let $\Omega_e = \{x \in \mathbb{R}^2 : \|x\| > 1\}$. Show that the Poincarè inequality does not hold in $H_0^1(\Omega_e)$, i.e., there does not exist a constant $c > 0$ satisfying

$$c\|u\|^2_{L^2(\Omega_e)} \le \int_{\Omega_e} \|\nabla u\|^2 \, dx \quad \text{for all } u \in H_0^1(\Omega_e). \tag{7}$$

The space $H_0^1(\Omega_e)$ is the completion of $C_0^\infty(\Omega_e)$ in the norm

$$\|v\|_{H^1(\Omega_e)} = \left(\|v\|^2_{L^2(\Omega_e)} + \|\nabla v\|^2_{(L^2(\Omega_e))^2}\right)^{1/2}. \tag{8}$$

(Hint: Consider dilating a fixed function.)

**Proof:** Consider the rotationally symmetric bump function,

$$\phi(r,\theta) := \phi(r) := \begin{cases} \exp(\frac{-1}{r(1-r)}) & \text{for } 0 < r < 1 \\ 0 & \text{for } r \ge 1, \end{cases}$$

which is only a function of $r$. Consider the sequence of bump functions defined by,

$$\phi_n(r) := \phi\Big(\frac{r-1}{n}\Big).$$

Note that $\text{supp}(\phi_n) = [1, n+1]$. Then consider the $L^2$ norm of $\phi_n$,

$$\begin{aligned} \|\phi_n\|^2_{L^2(\Omega_e)} &= \int_0^{2\pi} \int_1^\infty \phi_n^2(r) r \, dr \, d\theta \\ &= 2\pi \int_1^{n+1} \phi^2(\frac{r-1}{n}) r \, dr \\ &= 2\pi \int_0^1 \phi^2(s) n(ns+1) \, ds \\ &\ge 2\pi n^2 \int_0^1 s\phi^2(s) \, ds \\ &= C_0 n^2. \end{aligned}$$

Thus $\lim_{n\to\infty} \|\phi_n\|_{L^2(\Omega_e)} = \infty$. But consider the $H^1$-semi norm of $\phi_n$.

$$\begin{aligned} |\phi_n|^2_{H^1(\Omega_e)} &= \int_0^{2\pi} \int_1^\infty |\phi_n'(r)|^2 r \, dr \, d\theta \\ &= 2\pi \int_1^{n+1} \Big|\frac{\partial}{\partial r}\phi(\frac{r-1}{n})\Big|^2 r \, dr \\ &= 2\pi \int_0^1 \Big|\frac{1}{n}\frac{\partial}{\partial s}\phi(s)\Big|^2 n(ns+1) \, ds \\ &\le 2\pi \frac{n(n+1)}{n^2} \int_0^1 |\phi'(s)|^2 \, ds \\ &\le C_1 \end{aligned}$$

Thus since the $H^1$-semi norm is bounded for all $n$ and the $L^2$ norm blows up to infinity, this implies that we cannot have a Poincarè inequality on this domain. ∎