Applied/Numerical Qualifier Solution: August 2011

Bennett Clayton

Texas A&M University

August 1, 2021

Problem 1. Let $\mathbb{P}_2$ be the space of polynomials in two variables spanned by

$$\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}, \tag{1}$$

let $\hat{T}$ be the reference unit triangle, $\hat{\gamma}$ one side of $\hat{T}$, and $\hat{\pi}$ the standard Lagrange interpolant in $\hat{T}$ with values in $\mathbb{P}_2$.

Recall that there exists a constant C only depending on the geometry of $\hat{T}$ such that $\forall v \in H^3(\hat{T})$,

$$\inf_{p\in\mathbb{P}_2} ||v + p||_{H^3(\hat{T})} \leq C|v|_{H^3(\hat{T})} \tag{2}$$

a. State the trace theorem relating $L^2(\hat{\gamma})$ and $H^1(\hat{T})$.

Solution: The trace theorem in general says: For Ω a bounded domain and $\partial\Omega$ a Lipschitz boundary, there exists a bounded linear map $B : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that

- $Bu = u|_{\partial\Omega}, \quad \forall u \in W^{1,p}(\Omega)$

- $||Bu||_{L^2(\partial\Omega)} \leq C_{p,\Omega}||u||_{W^{1,p}(\Omega)}$

for some constant $C_{p,\Omega}$ depending on p and Ω.

b. Prove that there exists a constant $\hat{C}$ only depending on the geometry of $\hat{T}$ and $\hat{\gamma}$ such that $\forall \hat{u} \in H^3(\hat{T})$,

$$||\hat{u} + \hat{\pi}(\hat{u})||_{L^2(\hat{\gamma})} \leq \hat{C}|\hat{u}|_{H^3(\hat{T})} \tag{3}$$

Proof: Note since $\hat{u}$ and $\hat{\pi}(\hat{u})$ are in $H^3(\widehat{T}) \subset H^1(\widehat{T})$ we can apply the trace inequality

$$\begin{aligned}\|\hat{u} - \hat{\pi}(\hat{u})\|_{L^2(\hat{\gamma})} &\leq \|\hat{u} - \hat{\pi}(\hat{u})\|_{L^2(\partial\widehat{T})} \\ &\leq C_{2,\hat{T}} \|\hat{u} - \hat{\pi}(\hat{u})\|_{H^1(\widehat{T})}\end{aligned}$$

Next, note that for all $p \in \mathbb{P}_2$ we have $\hat{\pi}(p) = p$. In addition, notice that $\|(\mathrm{Id} - \hat{\pi})(\hat{u})\|_{H^1(\widehat{T})}$ is a bounded sublinear functional on $H^3(\widehat{T})$ that is zero for all polynomials in $\mathbb{P}^2$. So by the Bramble-Hilbert lemma, we have,

$$C_{2,\hat{T}} \|(\mathrm{Id} - \hat{\pi})(\hat{u})\|_{H^1(\hat{T})} \leq \hat{C} |\hat{u}|_{H^3(\hat{T})} \tag{4}$$

Thus, the result follows.

c. Let

$$X_h = \{v_h \in C^0(\overline{\Omega}) \,:\, \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_2\}. \tag{5}$$

Let T be a triangle of $\mathcal{T}_h$ with diameter h_T and diameter of inscribed disc ϱ_T, and let γ be one side of T. Let F_T be the affine mapping from $\hat{T}$ onto T and let $\pi_{2,h}$ denote the standard Lagrange interpolant on X_h. Prove that there exists a constant C only depending on the geometry of $\hat{T}$ and γ such that $\forall u \in H^3(T)$,

$$\|u - \pi_{2,h}(u)\|_{L^2(\gamma)} \leq C \sigma_T h_T^{2+1/2} |u|_{H^3(T)}, \tag{6}$$

where $\sigma = h_T/\varrho_T$.

Proof: Consider,

$$\begin{aligned}\|u - \pi_{2,h}(u)\|^2_{L^2(\gamma)} &= \int_\gamma |u - \pi_{2,h}(u)|^2 \, ds \\ &= \int_{\hat{\gamma}} |\hat{u} - \hat{\pi}_{2,h}(\hat{u})|^2 \frac{|\gamma|}{|\hat{\gamma}|} \, d\hat{s} \\ &\leq h_T \int_{\hat{\gamma}} |\hat{u} - \hat{\pi}_{2,h}(\hat{u})|^2 \, d\hat{s} \\ &= h_T \|\hat{u} - \hat{\pi}_{2,h}(\hat{u})\|^2_{L^2(\hat{\gamma})} \\ &\leq C h_T |\hat{u}|^2_{H^3(\hat{T})}.\end{aligned}$$

Now to change to the element T, let the affine geometric map F_T be defined as $F_T(\hat{\boldsymbol{x}}) = \mathsf{B}\hat{\boldsymbol{x}} + \boldsymbol{b}$. Then note that there exists a constant c independent of h_T such that $\|\mathsf{B}\| \leq c h_T$, with $\|\cdot\|$ be some arbitrary matrix norm. We make a change of variables through the geometric mapping, that is, we define $u := \hat{u} \circ F_T^{-1}$. Recall the H^3 semi-norm is given by,

$$|u|^2_{H^3(T)} = \int_T \sum_{|\alpha|=3} \left|(D^\alpha u)(\boldsymbol{x})\right|^2 d\boldsymbol{x}, \tag{7}$$

where $\alpha = (\alpha_1, \alpha_2)$ is the multi-index, $|\alpha| = \alpha_1 + \alpha_2$, and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial_x^{\alpha_1} \partial_y^{\alpha_2}}$. From calculus and based on our definition of the affine geometric mapping F_T, we claim there exists a constant, c_{B} such that $|(\widehat{D}^\alpha \hat{u})(\hat{\boldsymbol{x}})| \leq c h_T^3 |(D^\alpha u)(\boldsymbol{x})|$ for $|\alpha| = 3$. This result could be (tediously) proven by explicitly writing the transformation, F_T, applying the chain rule to each derivative, pulling out the factor h_T, and then recombining everything back. To summarize, we have the following,

$$
\begin{aligned}
|\hat{u}|^2_{H^3(\widehat{T})} &= \int_{\widehat{T}} \sum_{|\alpha|=3} |(\hat{D}^\alpha \hat{u})(\hat{\boldsymbol{x}})|^2 \, d\hat{\boldsymbol{x}} \\
&\leq \int_T c h_T^6 \sum_{|\beta|=3} |(D^\beta u)(\boldsymbol{x})|^2 |\det(F_T')| \, d\boldsymbol{x} \\
&= \int_T c h_T^6 \sum_{|\beta|=3} |(D^\beta u)(\boldsymbol{x})|^2 \frac{|\widehat{T}|}{|T|} \, d\boldsymbol{x} \\
&= c h_T^6 \frac{|\widehat{T}|}{|T|} |u|^2_{H^3(T)}.
\end{aligned}
$$

We know that $|T| \leq \frac{1}{2} h_T^2$, however, we need an approximation of the area with respect to the inscribed circle. We can prove that

$$
|T| = \frac{1}{2}(a + b + c)\frac{\varrho_T}{2}, \tag{8}
$$

where a, b, and c are the side lengths of our triangle T, by subdividing the T into three small triangles formed from the center of the inscribed circle. Then we have the following inequalities,

$$
\frac{1}{2} h_T \varrho_T = \frac{1}{2}(2h_T)\frac{\varrho_T}{2} \leq |T| \leq \frac{1}{2}(3h_T)\frac{\varrho_T}{2} = \frac{3}{4} h_T \varrho_T,
$$

where we have used the fact that the sum of any two sides of the triangle is greater than h_T. In particular, we can say that $|T|^{-1} \leq 2 h_T^{-1} \varrho_T^{-1}$. Using this fact, we finish the proof as follows,

$$
\begin{aligned}
\|u - \pi_{2,h}(u)\|^2_{L^2(\gamma)} &\leq C h_T \int_{\hat{T}} \sum_{|\alpha|=3} |(\hat{D}^\alpha \hat{u})(\hat{\boldsymbol{x}})|^2 \, d\hat{\boldsymbol{x}} \\
&= C h_T |\hat{u}|^2_{H^3(\widehat{T})} \\
&\leq C h_T h_T^6 \frac{|\widehat{T}|}{|T|} |u|^2_{H^3(T)} \\
&= C \frac{h_T}{\varrho_T} h_T^5 |u|^2_{H^3(T)}.
\end{aligned}
$$

Taking the square root we have,

$$
\|u - \pi_{2,h}(u)\|_{L^2(\gamma)} \leq C \sigma_T^{1/2} h_T^{2+1/2} |u|_{H^3(T)}.
$$

■

Problem 2. Let $\delta > 0$ be a given constant parameter and $u \in H_0^1(\Omega)$ a given function. Consider the problem: Find $\varphi^\delta \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\delta^2 \Delta \varphi^\delta(x) + \varphi^\delta(x) &= u(x) \text{ a.e. in } \Omega, \\ \varphi^\delta(x) &= 0 \text{ a.e. on } \partial\Omega. \end{aligned} \tag{9}$$

a. Define the bilinear form

$$a_\delta(w, v) = \delta^2 \int_\Omega \nabla w(x) \cdot \nabla v(x)\, dx + \int_\Omega w(x) v(x)\, dx. \tag{10}$$

Write the variational formulation of Problem (9) and prove that it has one and only one solution $\varphi^\delta \in H_0^1(\Omega)$.

Proof: Define $f_u(v) := \int_\Omega u(x)v(x)\, dx$. Then the variational problem is: Find $\varphi^\delta \in H_0^1(\Omega)$ such that

$$a_\delta(\varphi^\delta, v) = f_u(v)$$

for all $v \in H_0^1(\Omega)$. To show that there is one and only one solution, we can use Lax-Milgram theorem. We first need to show that a_δ and f_u are both continuous and a_δ is coercive. So consider,

$$\begin{aligned} a_\delta(w, v) &= \delta^2 \int_\Omega \nabla w(x) \cdot \nabla v(x)\, dx + \int_\Omega w(x) v(x)\, dx \\ &\le \delta^2 |w|_{H^1(\Omega)} |v|_{H^1(\Omega)} + \|w\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\le \max\{\delta^2, 1\} \big(|w|_{H^1(\Omega)} |v|_{H^1(\Omega)} + \|w\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \big) \\ &\le \max\{\delta^2, 1\} \|w\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned}$$

Thus a_δ is continuous. Cauchy-Schwarz shows that f_u is continuous, so we only need to check that a_δ is coercive. Consider,

$$\begin{aligned} a_\delta(v, v) &= \delta^2 \int_\Omega |\nabla v|^2\, dx + \int_\Omega v^2\, dx \\ &\ge \min\{\delta^2, 1\} \|v\|_{H^1(\Omega)}^2. \end{aligned}$$

Thus by Lax-Milgram, there exists a unique solution to the variational formulation. ■

b. Prove that

$$\|\varphi^\delta\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}. \tag{11}$$

Proof: From the variational equation, we have

$$a(\varphi^\delta, \varphi^\delta) = f_u(\varphi^\delta).$$

Hence we can write,

$$\|\varphi^\delta\|_{L^2(\Omega)}^2 \leq \int_\Omega \delta^2 |\nabla\varphi^\delta|^2 + (\varphi^\delta)^2 \, dx = \int_\Omega u\varphi^\delta \, dx \leq \|u\|_{L^2(\Omega)} \|\varphi^\delta\|_{L^2(\Omega)}.$$

Dividing by $\|\varphi^\delta\|_{L^2(\Omega)}$ we have the result. ∎

c. Prove that

$$\|\nabla\varphi^\delta\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}. \tag{12}$$

Hint: observe that $\Delta\varphi^\delta$ belongs to $L^2(\Omega)$, take the scalar product of (9) with $-\Delta\varphi^\delta$ and apply Green's formula.

Proof: To see that $\Delta\varphi^\delta$ is in $L^2(\Omega)$, consider,

$$\|\delta^2 \Delta\varphi^\delta\|_{L^2(\Omega)} = \|u - \varphi^\delta\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} + \|\varphi^\delta\|_{L^2(\Omega)} < \infty.$$

So, following the hint, we have,

$$\begin{aligned}
\int_\Omega \delta^2 (\Delta\varphi^\delta)^2 - \varphi^\delta \Delta\varphi^\delta \, dx &= \delta^2 \int_\Omega (\Delta\varphi^\delta)^2 \, dx + \int_\Omega |\nabla\varphi^\delta|^2 \, dx - \int_{\partial\Omega} \varphi^\delta \frac{\partial\varphi^\delta}{\partial n} \, ds \\
&= \delta^2 \int_\Omega (\Delta\varphi^\delta)^2 \, dx + \int_\Omega |\nabla\varphi^\delta|^2 \, dx \\
&= \delta^2 \|\Delta\varphi^\delta\|_{L^2(\Omega)}^2 + \|\nabla\varphi^\delta\|_{L^2(\Omega)}^2
\end{aligned}$$

For the right hand side, we have,

$$-\int_\Omega u\Delta\varphi^\delta \, dx = \int_\Omega \nabla u \cdot \nabla\varphi^\delta \, dx - \int_{\partial\Omega} u \frac{\partial\varphi^\delta}{\partial n} \, ds = \int_\Omega \nabla u \cdot \nabla\varphi^\delta \, dx$$

Hence we have $\delta^2 \|\Delta\varphi^\delta\|_{L^2(\Omega)}^2 + \|\nabla\varphi^\delta\|_{L^2(\Omega)}^2 = \int_\Omega \nabla u \cdot \nabla\varphi^\delta \, dx$. Using the usual inequality tricks, we have,

$$\|\nabla\varphi^\delta\|_{L^2(\Omega)}^2 \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla\varphi^\delta\|_{L^2(\Omega)}.$$

Dividing by $\|\nabla\varphi^\delta\|_{L^2(\Omega)}$ we thus have the result.

d. Now let

$$X_{0,h} = \{v_h \in C^0(\Omega) \, : \, \forall T \in \mathcal{T}_h, \, v_h|_T \in \mathbb{P}_1, \, v_h|_{\partial\Omega} = 0\}. \tag{13}$$

Given $u_h \in X_{0,h}$, consider the discrete problem: Find $\varphi_h^\delta \in X_{0,h}$, satisfying $\forall v_h \in X_{0,h}$,

$$a_\delta(\varphi_h^\delta, v_h) = \int_\Omega u_h(x) v_h(x)\, dx. \tag{14}$$

(i) Prove that problem (14) has one and only one solution $\varphi_h^\delta \in X_{0,h}$.

(ii) Prove that

$$\|\varphi_h^\delta\|_{L^2(\Omega)} \le \|u_h\|_{L^2(\Omega)}. \tag{15}$$

Proof: For part (i) note that $X_{0,h}$ is a subspace of $H_0^1(\Omega)$, then Lax-Milgram applies in this case. For part (ii) the same method we used in part b. can be applied again here, while taking care with the gradient since $X_{0,h}$ consists of piecewise linear functions. ■

e. Assume that φ^δ belongs to $H^2(\Omega)$. Let $\pi_{1,h}$ denote the standard Lagrange interpolant on $X_{0,h}$.

(i) Prove that

$$a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \varphi_h^\delta) = a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) - \int_\Omega (u - u_h)(\varphi_h^\delta - \varphi^\delta + \varphi^\delta - \pi_{1,h}(\varphi^\delta))\, dx.$$

(ii) Assuming that u is smooth enough, $u_h = \pi_{1,h}(u)$, and $\delta = h$, derive an estimate for $\|\varphi^\delta - \varphi_h^\delta\|_{L^2(\Omega)}$.

Proof: For part (i) consider,

$$\begin{aligned}
a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \varphi_h^\delta) &= a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta) + \pi_{1,h}(\varphi^\delta) - \varphi_h^\delta) \\
&= a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) + a_\delta(\varphi^\delta, \pi_{1,h}(\varphi^\delta) - \varphi_h^\delta) - a_\delta(\varphi_h^\delta, \pi_{1,h}(\varphi^\delta) - \varphi_h^\delta) \\
&= a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) + \int_\Omega u(\pi_{1,h}(\varphi^\delta) - \varphi_h^\delta)\, dx \\
&\qquad - \int_\Omega u_h(\pi_{1,h}(\varphi^\delta) - \varphi_h^\delta)\, dx \\
&= a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) + \int_\Omega (u - u_h)(\varphi_h^\delta - \pi_{1,h}(\varphi^\delta))\, dx \\
&= a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) + \int_\Omega (u - u_h)(\varphi_h^\delta - \varphi^\delta + \varphi^\delta - \pi_{1,h}(\varphi^\delta))\, dx.
\end{aligned}$$

Note we have used the fact that $\pi_{1,h}(\varphi^\delta) - \varphi_h^\delta \in X_{0,h}$, so φ^δ and φ_h^δ will solve their respective variational equations.

Now for part (ii). To simplify notation, set $\|\varphi^\delta - \varphi_h^\delta\|_h^2 := h^2|\varphi^\delta - \varphi_h^\delta|_{H^1(\Omega)}^2 + \|\varphi^\delta - \varphi_h^\delta\|_{L^2(\Omega)}^2$. Using the identity we proved in part (i) and continuity and coercivity of a, we have,

$$\begin{aligned}
\|\varphi^\delta - \varphi_h^\delta\|_h^2 &= h^2|\varphi^\delta - \varphi_h^\delta|_{H^1(\Omega)}^2 + \|\varphi^\delta - \varphi_h^\delta\|_{L^2(\Omega)}^2 \\
&= a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \varphi_h^\delta) \\
&= a_\delta(\varphi^\delta - \varphi_h^\delta, \varphi^\delta - \pi_{1,h}(\varphi^\delta)) + \int_\Omega (u - u_h)(\varphi_h^\delta - \varphi^\delta + \varphi^\delta - \pi_{1,h}(\varphi^\delta))\, dx \\
&\le \|\varphi^\delta - \varphi_h^\delta\|_{L^2(\Omega)}\|\varphi^\delta - \pi_{1,h}(\varphi^\delta)\|_{L^2(\Omega)} + h^2|\varphi^\delta - \varphi_h^\delta|_{H^1(\Omega)}|\varphi^\delta - \pi_{1,h}(\varphi^\delta)|_{H^1(\Omega)} \\
&\quad + \|u - u_h\|_{L^2(\Omega)}(\|\varphi_h^\delta - \varphi^\delta\|_{L^2(\Omega)} + \|\varphi^\delta - \pi_{1,h}(\varphi^\delta)\|_{L^2(\Omega)}), \\
&\le \frac{1}{2}\|\varphi^\delta - \varphi_h^\delta\|_{L^2(\Omega)}^2 + \frac{h^2}{2}|\varphi^\delta - \varphi_h^\delta|_{H^1(\Omega)}^2 \\
&\quad + \frac{1}{2}\|\varphi^\delta - \pi_{1,h}(\varphi^\delta)\|_{L^2(\Omega)}^2 + \frac{h^2}{2}|\varphi^\delta - \pi_{1,h}(\varphi^\delta)|_{H^1(\Omega)}^2 \\
&\quad + \|u - u_h\|_{L^2(\Omega)}(\|\varphi_h^\delta - \varphi^\delta\|_{L^2(\Omega)} + \|\varphi^\delta - \pi_{1,h}(\varphi^\delta)\|_{L^2(\Omega)}).
\end{aligned}$$

Subtracting $\frac{1}{2}\|\varphi^\delta - \varphi_h^\delta\|_{L^2(\Omega)}^2 + \frac{h^2}{2}|\varphi^\delta - \varphi_h^\delta|_{H^1(\Omega)}^2$ from both sides of the inequality, we find,

$$\begin{aligned}
\frac{1}{2}\|\varphi^\delta - \varphi_h^\delta\|_h^2 \le \frac{1}{2}\|\varphi^\delta - \pi_{1,h}(\varphi^\delta)\|_{L^2(\Omega)}^2 + \frac{h^2}{2}|\varphi^\delta - \pi_{1,h}(\varphi^\delta)|_{H^1(\Omega)}^2 \\
+ \|u - u_h\|_{L^2(\Omega)}(\|\varphi_h^\delta - \varphi^\delta\|_{L^2(\Omega)} + \|\varphi^\delta - \pi_{1,h}(\varphi^\delta)\|_{L^2(\Omega)}).
\end{aligned} \tag{16}$$

The goal now is to apply the Bramble-Hilbert lemma to the norms with the projection operators. See the older exams for more fleshed out proofs on how to apply the Bramble-Hilbert lemma. It is also worth mentioning that we cannot use Ceà's lemma since this applies to the space $(H_0^1(\Omega), \|\cdot\|_{H^1(\Omega)})$ and would therefore give us the wrong error estimate.

Continuing on, we have,

$$\begin{aligned}
\frac{1}{2}\|\varphi^\delta - \varphi_h^\delta\|_h^2 &\le Ch^4|\varphi^\delta|_{H^2(\Omega)}^2 + Ch^2|u|_{H^2(\Omega)}\big(\|\varphi_h^\delta - \varphi^\delta\|_{L^2(\Omega)} + Ch^2|\varphi^\delta|_{H^2(\Omega)}\big). \\
&\le Ch^4|\varphi^\delta|_{H^2(\Omega)}^2 + Ch^4|u|_{H^2(\Omega)}^2 + \frac{1}{4}\|\varphi_h^\delta - \varphi^\delta\|_{L^2(\Omega)}^2 + Ch^4|u|_{H^2(\Omega)}|\varphi^\delta|_{H^2(\Omega)}.
\end{aligned}$$

In the last inequality, we used the inequality $ab \le \frac{1}{2}(a^2 + b^2)$, but in the form,

$$Ch^2|u|_{H^2(\Omega)}\|\varphi_h^\delta - \varphi^\delta\|_{L^2(\Omega)} = \big(\sqrt{2}Ch^2|u|_{H^2(\Omega)}\big)\big(\frac{1}{\sqrt{2}}\|\varphi_h^\delta - \varphi^\delta)\|_{L^2(\Omega)}\big).$$

Finally, we can drop the H^1-seminorm on the left hand side, to find that,

$$\frac{1}{4}\|\varphi^\delta - \varphi_h^\delta\|_{L^2(\Omega)}^2 \le Ch^4(|\varphi^\delta|_{H^2(\Omega)}^2 + |u|_{H^2(\Omega)}^2 + |u|_{H^2(\Omega)}|\varphi^\delta|_{H^2(\Omega)}),$$

which reduces to,

$$\|\varphi^\delta - \varphi_h^\delta\|_{L^2(\Omega)} \le Ch^2(|\varphi^\delta|_{H^2(\Omega)} + |u|_{H^2(\Omega)}).$$

■

Problem 3. Let $T > 0$ be a given final time, let $\vec{b}$ be a given vector valued function with components in $L^2(0,T;H^1(\Omega)) \cap C^0(\Omega \times [0,T])$ and let u_0 be a given real valued function in $C^0(\Omega)$. We suppose that

$$\text{div } \vec{b} = 0 \text{ a.e. in } \Omega, \vec{b} = \vec{0} \text{ on } \Gamma. \tag{17}$$

Consider the time-dependent problem: Find u such that

$$\begin{aligned} \frac{\partial u}{\partial t}(\boldsymbol{x},t) + \vec{b}(\boldsymbol{x},t) \cdot \nabla u(\boldsymbol{x},t) = 0 \text{ a.e. in } \Omega \times (0,T), \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \text{ a.e. in } \Omega, \end{aligned} \tag{18}$$

where

$$\vec{b} \cdot \nabla u = b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} \tag{19}$$

Accept as a fact that (18) has one and only one solution u that is sufficiently smooth. It is discretized as follows in space and time. Let

$$X_h = \{v_h \in C^0(\Omega) : \forall T \in \mathcal{T}_h, v_h|_T \in \mathbb{P}_1\}. \tag{20}$$

Choose an integer $K \geq 2$, set $k = T/K$, $t_n = nk$ and $u_h^0 = \pi_{1,h}(u_0)$. For $1 \leq n \leq K$, define $u_h^n \in X_h$ from u_h^{n-1} recursively by,

$$\frac{1}{k}\int_\Omega (u_h^n - u_h^{n-1})(\boldsymbol{x}) v_h(\boldsymbol{x})\, d\boldsymbol{x} + \int_\Omega (\vec{b}(\boldsymbol{x},t_n) \cdot \nabla u_h^n(\boldsymbol{x})) v_h(\boldsymbol{x})\, d\boldsymbol{x} = 0, \tag{21}$$

for all $v_h \in X_h$.

a.) Prove that

$$\int_\Omega (\vec{b}(\boldsymbol{x},t_n) \cdot \nabla v_h(\boldsymbol{x})) v_h(\boldsymbol{x})\, d\boldsymbol{x} = 0, \tag{22}$$

for all $v_h \in X_h$.

Proof: To prove this, we will use integration by parts. So consider,

$$\begin{aligned} \int_\Omega (\vec{b}(\boldsymbol{x},t_n) \cdot \nabla v_h(\boldsymbol{x})) v_h(\boldsymbol{x})\, d\boldsymbol{x} &= \int_\Omega (v_h(\boldsymbol{x})\vec{b}(\boldsymbol{x},t_n)) \cdot \nabla v_h(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int_{\partial\Omega} v_h^2(\boldsymbol{x})\vec{b}(\boldsymbol{x},t_n) \cdot \mathbf{n}\, ds - \int_\Omega \nabla \cdot (v_h(\boldsymbol{x})\vec{b}(\boldsymbol{x},t_n)) v_h(\boldsymbol{x})\, d\boldsymbol{x} \\ &= -\int_\Omega \Big(\frac{\partial}{\partial \boldsymbol{x}_1}(v_h(\boldsymbol{x}) b_1(\boldsymbol{x},t_n)) + \frac{\partial}{\partial \boldsymbol{x}_2}(v_h(\boldsymbol{x}) b_2(\boldsymbol{x},t_n))\Big) v_h(\boldsymbol{x})\, d\boldsymbol{x} \\ &= -\int_\Omega \Big(\nabla v_h(\boldsymbol{x}) \cdot \vec{b}(\boldsymbol{x},t_n) + v_h(\boldsymbol{x}) \nabla \cdot \vec{b}(\boldsymbol{x},t)\Big) v_h(\boldsymbol{x})\, d\boldsymbol{x} \\ &= -\int_\Omega (\vec{b}(\boldsymbol{x},t_n) \cdot \nabla v_h(\boldsymbol{x})) v_h(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned}$$

Thus, adding the right hand side to the left side, we have the result. ■

b. Show that, given $u_h^{n-1} \in X_h$, (21) has one and only one solution $u_h^n \in X_h$.

Proof: Let $\{\phi_i\}_{i=1}^N$ be the nodal basis for X_h. Then we write $u_h^n(\boldsymbol{x}) = \sum_{i=1}^N u_i^n \phi_i(\boldsymbol{x})$. Using this expansion for u_h^n, we test (21) with $v_h = \phi_j$ for $j = 1, \dots, N$. Doing this, we have,

$$\frac{1}{k}\sum_{i=1}^{N}(u_i^n - u_i^{n-1})\int_\Omega \phi_i(\boldsymbol{x})\phi_j(\boldsymbol{x})\,d\boldsymbol{x} + \sum_{i=1}^{N} u_i^n \int_\Omega \vec{b}(\boldsymbol{x}, t_n)\cdot\nabla\phi_i(\boldsymbol{x})\phi_j(\boldsymbol{x})\,d\boldsymbol{x} = 0 \tag{23}$$

This N equations can be viewed as a matrix equation,

$$\frac{1}{k}\mathsf{M}(\mathbf{U}^n - \mathbf{U}^{n-1}) + \mathsf{B}\mathbf{U}^n = \mathbf{0}, \tag{24}$$

where M is the matrix with entries, $\int_\Omega \phi_i\phi_j\,d\boldsymbol{x}$, B is the matrix with entries, $\int_\Omega(\vec{b}\cdot\nabla\phi_i)\phi_j\,d\boldsymbol{x}$, and $\mathbf{U}^n = (u_1^n, \dots, u_N^n)^T$. This equation can be written in the standard matrix equation form,

$$(\mathsf{M} + k\mathsf{B})\mathbf{U}^n = \mathsf{M}\mathbf{U}^{n-1}. \tag{25}$$

The question of existence and uniqueness is now framed in terms of a matrix equation. That is, we know a solution exists and is unique if and only if, nullity$(\mathsf{M}+k\mathsf{B}) = 0$ and rank$(\mathsf{M}+k\mathsf{B}) = N$.

Assume that $\mathbf{U}^{n-1} = \mathbf{0}$, we wish to prove that $\mathbf{U}^n = \mathbf{0}$ is the only solution. This is equivalent to proving that the only solution to $\frac{1}{k}(u_h^n, v_h) + \int_\Omega(\vec{b}\cdot\nabla u_h^n)v_h\,d\boldsymbol{x} = 0$ is $u_h^n \equiv 0$. Test this equation with $v_h = u_h^n$ and we find $\frac{1}{k}\|u_h^n\|_{L^2(\Omega)}^2 + \int_\Omega(\vec{b}\cdot\nabla u_h^n)u_h^n\,d\boldsymbol{x} = \frac{1}{k}\|u_h^n\|_{L^2(\Omega)}^2 = 0$. Therefore the only solution is $u_h^n = 0$; i.e. nullity$(\mathsf{M} + k\mathsf{B}) = 0$. Additionally, the matrix is square, so by the rank-nullity theorem, rank$(\mathsf{M} + k\mathsf{B}) = N$. This completes the proof. ∎

c. Prove for $1 \le n \le K$

$$\|u_h^n\|_{L^2(\Omega)} \le \|u_h^0\|_{L^2(\Omega)}. \tag{26}$$

Proof: From equation (21), take $v_h = u_h^n$. Then by part a. we have,

$$\frac{1}{k}(u_h^n - u_h^{n-1}, u_h^n) = 0.$$

Using Cauchy-Schwarz inequality, we have

$$\|u_h^n\|_{L^2(\Omega)}^2 \le \|u_h^n\|_{L^2(\Omega)}\|u_h^{n-1}\|_{L^2(\Omega)}.$$

Dividing by $\|u_h^n\|_{L^2(\Omega)}$ we then have,

$$\|u_h^n\|_{L^2(\Omega)} \le \|u_h^{n-1}\|_{L^2(\Omega)} \le \dots \le \|u_h^0\|_{L^2(\Omega)}.$$

■

d. Is the matrix of the system (21) symmetric ? Justify your answer.

Proof: No the matrix will not be symmetric. This is due to the term,

$$a(u_h, v_h) := \int_\Omega (\vec{b}(\boldsymbol{x}, t_n) \cdot \nabla u_h) v_h(\boldsymbol{x})\, d\boldsymbol{x},$$

which is not a symmetric bilinear form. To see this, we can apply integration by parts (as we did in part a.) to get

$$\begin{aligned} a(u_h, v_h) &= -\int_\Omega \nabla \cdot (v_h(\boldsymbol{x})\vec{b}(\boldsymbol{x}, t_n)) u_h \, d\boldsymbol{x} \\ &= -\int_\Omega \vec{b}(\boldsymbol{x}, t_n) \cdot \nabla v_h(\boldsymbol{x}) u_h(\boldsymbol{x})\, d\boldsymbol{x} \\ &= -\int_\Omega \nabla \cdot (\vec{b}(\boldsymbol{x}, t_n) v_h(\boldsymbol{x})) u_h(\boldsymbol{x})\, d\boldsymbol{x} \\ &= -a(v_h, u_h), \end{aligned}$$

where we have used the fact that $\nabla \cdot \vec{b} = 0$. Since the other integral term is symmetric, the end result, we will be a matrix which is not symmetric. ■