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Problem 1. Consider the variational problem: find $u \in H^1(\Omega)$, such that $a(u,v) = L(v)$ for all $v \in H^1(\Omega)$, where $\Omega = (0,1) \times (0,1)$, Γ is its boundary, and

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \, dy + \int_0^1 u(s,0)v(s,0) \, ds \quad \text{and} \quad L(v) = \int_\Gamma gv \, ds. \tag{1}$$

Let $V_h \subset H^1(\Omega)$ be a finite dimensional space of conforming piece-wise linear finite elements (Courant triangles) over regular partition of Ω into triangles. For continuous v, w defined on $\tilde{\Gamma} \subset \Gamma$, let the bilinear form $\mathcal{Q}_{\tilde{\Gamma}}(v,w)$ come from the quadrature

$$\mathcal{Q}_{\tilde{\Gamma}}(v,w) = \sum_{e \subset \tilde{\Gamma}} \frac{|e|}{2}\left(v(P_1^e)w(P_1^e) + v(P_2^e)w(P_2^e)\right) \approx \int_{\tilde{\Gamma}} vw \, ds. \tag{2}$$

Here e is an edge of the triangulation of length $|e|$ with end points P_1^e and P_2^e. Consider the FEM: find $u_h \in V_h$ such that

$$a_h(u_h, v) = L_h(v), \ \forall v \in V_h, \tag{3}$$

where $a_h(u_h, v)$ and $L_h(v)$ are defined from $a(u_h, v)$ and $L(v)$ with the boundary integrals approximated using quadrature (2).

a. Derive the strong form to the problem (1).

Solution: Let $\hat{\Gamma} = \Gamma - \{y = 0\} \times (0,1)$. Then performing integration by parts and we have,

$$\begin{aligned}
a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx \, dy + \int_0^1 u(s,0)v(s,0) \, ds \\
&= \int_\Omega -\Delta u v \, dx \, dy + \int_\Gamma \frac{\partial u}{\partial n} v \, ds + \int_0^1 u(s,0)v(s,0) \, ds \\
&= \int_\Omega -\Delta u v \, dx \, dy + \int_{\hat{\Gamma}} \frac{\partial u}{\partial n} v \, ds + \int_0^1 -\frac{\partial u}{\partial y}(s,0)v(s,0) \, ds + \int_0^1 u(s,0)v(s,0) \, ds. \\
&= \int_\Omega -\Delta u v \, dx \, dy + \int_{\hat{\Gamma}} \frac{\partial u}{\partial n} v \, ds + \int_0^1 (u(s,0) - u_y(s,0))v(s,0) \, ds
\end{aligned}$$

Now, if we only consider $v \in H_0^1(\Omega)$, then this implies,

$$\int_\Omega -\Delta u v \, dx \, dy = 0.$$

By the Fundamental Theorem of Variational Calculus, if $\int_\Omega f\phi \, dx = 0$ for all $\phi \in C_c^\infty(\Omega)$, then $f \equiv 0$. Since $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ a similar result holds for all $v \in H_0^1(\Omega)$. Therefore $-\Delta u = 0$. Now, for the boundary conditions. Since $-\Delta u = 0$, we have

$$\int_{\hat{\Gamma}} \frac{\partial u}{\partial n} v \, ds + \int_0^1 (u(s,0) - u_y(s,0))v(s,0) \, ds = \int_\Gamma gv \, ds, \tag{4}$$

for all $v \in H^1(\Omega)$. Since this identity holds for all $v \in H^1(\Omega)$, we must have

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = g & \text{on } \hat{\Gamma} \\ \frac{\partial u}{\partial n} + u = g & \text{on } \Gamma - \hat{\Gamma}. \end{cases} \tag{5}$$

■

b. Prove that the bilinear form $a(u,v)$ is **coercive** on H^1.

Solution: In order to prove coercivity, we will first prove the following inequality,

$$||u||_{L^2(\Omega)}^2 \leq C\Big(\Big|\Big|\frac{\partial u}{\partial y}\Big|\Big|_{L^2(\Omega)}^2 + \int_0^1 u^2(s,0) \, ds\Big).$$

So consider,

$$\begin{aligned} ||u||_{L^2(\Omega)}^2 &= \int_0^1 \int_0^1 u^2(x,y) \, dx \, dy \\ &= \int_0^1 \int_0^1 \Big(\int_0^y \frac{\partial u}{\partial \eta}(x,\eta) \, d\eta + u(x,0)\Big)^2 \, dx \, dy \\ &\leq \int_{[0,1]^2} 2\Big(\int_0^y \frac{\partial u}{\partial \eta}(x,\eta) \, d\eta\Big)^2 + 2u^2(x,0) \, dx \, dy \\ &\leq 2 \int_0^1 \int_0^1 \Big|\frac{\partial u}{\partial y}\Big|^2 \, dx \, dy + 2 \int_0^1 u^2(s,0) \, ds. \end{aligned}$$

Note we have used the Cauchy-Schwarz inequality and the inequality, $(a+b)^2 \leq 2a^2 + 2b^2$. Now for the Poincarè inequality,

$$
\begin{aligned}
a(u,u) &= \int_{[0,1]^2} \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 dx\,dy + \int_0^1 u^2(s,0)\,ds \\
&\geq \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\Big(\int_{[0,1]^2} \left|\frac{\partial u}{\partial y}\right|^2 dx\,dy + \int_0^1 u^2(s,0)\,ds\Big) \\
&\geq \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega))} + \frac{1}{4}\|u\|^2_{L^2(\Omega)} \\
&\geq \frac{1}{4}\|u\|^2_{H^1(\Omega)}.
\end{aligned}
$$

Thus $a(\cdot,\cdot)$ is coercive on $H^1(\Omega)$. ■

c. Prove that for $\tilde{\Gamma} = \{(x,0), 0 < x < 1\}$, there are constants c_1 and c_2, independent of h, such that

$$c_1 \mathcal{Q}_{\tilde{\Gamma}}(v,v) \leq \int_0^1 v(x,0)^2\,dx \leq c_2 \mathcal{Q}_{\tilde{\Gamma}}(v,v),\ \forall v \in V_h. \tag{6}$$

Note that this inequality and part b. immediately imply

$$a_h(v,v) \geq \alpha \|v\|^2_{H^1(\Omega)},\ \forall v \in V_h \tag{7}$$

for some $\alpha > 0$ independent of h.

Proof: First note that $v|_e$ is linear and $(v|_e)^2$ will be a **convex** quadratic function. Therefore, the trapezoidal rule will give an upper estimate of $\int_0^1 v(x,0)^2\,dx$. So we have,

$$\int_0^1 v(x,0)^2\,dx = \sum_{e\subset\tilde{\Gamma}} \int_e v(x,0)^2\,dx \leq \sum_{e\subset\tilde{\Gamma}} \frac{|e|}{2}(v^2(P_1^e) + v^2(P_2^e)) = \mathcal{Q}_{\tilde{\Gamma}}(v,v)$$

For the lower bound, let $T_e : [-1,1] \to e$ be the usual affine transformation, and define $\hat{v}_e := v \circ T_e$. As before, $\hat{v}_e$ is linear, so we write $\hat{v}_e(\hat{x},0) = a_e\hat{x} + b_e$, then through direct computation, we have,

$$\int_{-1}^1 (a_e\hat{x} + b_e)^2\,d\hat{x} = \frac{1}{3}(2a_e^2 + 6b_e^2) \geq \frac{1}{3}(2a_e^2 + 2b_e^2).$$

But notice that $\hat{v}_e^2(1,0) + \hat{v}_e^2(-1,0) = 2a_e^2 + 2b_e^2$. Thus, we have that

$$\begin{aligned}\int_0^1 v(x,0)^2\,dx &= \sum_{e\subset\tilde{\Gamma}} \int_e v(x,0)^2\,dx \\ &= \sum_{e\subset\tilde{\Gamma}} \frac{|e|}{2} \int_{-1}^1 \hat{v}_e^2(\hat{x},0)\,d\hat{x} \\ &\geq \sum_{e\subset\tilde{\Gamma}} \frac{|e|}{2}\cdot\frac{1}{3}(\hat{v}_e^2(1,0) + \hat{v}_e^2(-1,0)) \\ &= \frac{1}{3}\sum_{e\subset\tilde{\Gamma}} \frac{|e|}{2}(v^2(P_1^e) + v^2(P_2^2)).\end{aligned}$$

Hence we can conclude that,

$$\frac{1}{3}\mathcal{Q}_{\tilde{\Gamma}}(v,v) \leq \int_0^1 v^2(x,0)\,dx.$$

■

d. Apply Strang's First Lemma to **estimate the error** in H^1-norm for the FEM (3). You may assume that g is as regular (smooth) as needed by your analysis and you can use (without proof) standard approximation properties for the finite element space V_h.

Solution: Recall the inequality for Strang's lemma,

$$\|u - u_h\|_{H^1(\Omega)} \leq c\Big[\inf_{v_h\in V_h}\big(\|u - v_h\|_{H^1(\Omega)} + \|a(v_h,\cdot) - a_h(v_h,\cdot)\|_{*,h}\big) + \|L - L_h\|_{*,h}\Big],$$

and in order to apply Strang's lemma, we must have that $a_h(\cdot,\cdot)$ is V_h-elliptic. But this result follows from part c. So note that $|L(v_h) - L_h(v_h)| = |\int_{\tilde{\Gamma}} v_h g ds - \mathcal{Q}_{\tilde{\Gamma}}(v_h,g)|$ by the assumption in the statement of the problem. Additionally, we have $|a(v_h,z_h) - a_h(v_h,z_h)| = |\int_{\tilde{\Gamma}} v_h z_h ds - \mathcal{Q}_{\tilde{\Gamma}}(v_h,z_h)|$, again by the statement of the problem. So lets now focus on $|a(v_h,z_h) - a_h(v_h,z_h)|$,

$$\begin{aligned}|a(v_h,z_h) - a_h(v_h,z_h)| &= \Big|\int_0^1 v_h(s,0)z_h(s,0)\,ds - \sum_{e\subset\tilde{\Gamma}}\frac{|e|}{2}\big((v_h z_h)(P_1^e) + (v_h z_h)(P_2^e)\big)\Big| \\ &= \Big|\sum_{e\subset\tilde{\Gamma}}\int_e v_h(s,0)z_h(s,0)\,ds - \frac{|e|}{2}\big((v_h z_h)(P_1^e) + (v_h z_h)(P_2^e)\big)\Big| \\ &\leq \sum_{e\subset\tilde{\Gamma}}\frac{|e|}{2}\Big|\int_{-1}^1 \hat{v}_h(\hat{s},0)\hat{z}_h(\hat{s},0)\,d\hat{s} - ((\hat{v}_h\hat{z}_h)(-1,0) + (\hat{v}_h\hat{z}_h)(1,0))\Big|\end{aligned}$$

Define the sublinear functional, $E: H^2(-1,1) \to \mathbb{R}$ by,

$$E(v) := \Big|\int_{-1}^1 v\,ds - (v(-1) + v(1))\Big|.$$

We show that E is bounded,

$$\begin{aligned}|E(v)| &\leq \int_{-1}^{1} |v|\, ds + |v(-1)| + |v(1)| \\ &\leq \sqrt{2}\|v\|_{L^2(-1,1)} + \sqrt{2}\|v\|_{\ell^2(-1,1)} \\ &\leq \sqrt{2}(\|v\|_{L^2(-1,1)} + C_{\text{trace}}\|v\|_{H^1(-1,1)}) \\ &\leq C\|v\|_{H^2(-1,1)},\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and the trace theorem with $\|v\|_{\ell^2(-1,1)} := \sqrt{v(-1)^2 + v(1)^2}$. One can also verify that $E(p) = 0$ for all $p \in \mathbb{P}_1$. So by the Bramble-Hilbert lemma, we have $|E(u)| \leq C|u|_{H^2(-1,1)}$. For our problem, we have $|E(\hat{v}_h \hat{z}_h)| \leq C|\hat{v}_h \hat{z}_h|_{H^2(-1,1)}$. Applying this in our inequality, we have,

$$\begin{aligned}|a(v_h, z_h) - a_h(v_h, z_h)| &\leq C \sum_{e \subset \tilde{\Gamma}} \frac{|e|}{2} |\hat{v}_h \hat{z}_h|_{H^2(-1,1)} \\ &\leq Ch^{2+1/2} \sum_{e \subset \tilde{\Gamma}} |v_h z_h|_{H^2(e)} \\ &\leq Ch^{2+1/2} \sum_{e \subset \tilde{\Gamma}} \Big(\int_e \big((v_h z_h)''(x,0)\big)^2\, dx\Big)^{1/2} \\ &= Ch^{2+1/2} \sum_{e \subset \tilde{\Gamma}} \Big(\int_e 4(v_h'(x,0) z_h'(x,0))^2\, dx\Big)^{1/2}.\end{aligned}$$

where the prime $'$ notation is differentiation with respect to x. Since $\hat{v}_h$ and $\hat{z}_h$ are linear on $(-1,1)$, their derivatives are constant. Therefore, we can write,

$$\begin{aligned}|a(v_h, z_h) - a_h(v_h, z_h)| &\leq Ch^{2+1/2} \sum_{e \subset \tilde{\Gamma}} |v_h|_{H^1(e)} |z_h|_{H^1(e)} \\ &\leq Ch^{2+1/2} \Big(\sum_{e \subset \tilde{\Gamma}} |v_h|^2_{H^1(e)}\Big)^{1/2} \Big(\sum_{e \subset \tilde{\Gamma}} |z_h|^2_{H^1(e)}\Big)^{1/2} \\ &\leq Ch^{2+1/2} \Big(\sum_{e \subset \tilde{\Gamma}} \|\nabla v_h\|^2_{L^2(e)}\Big)^{1/2} \Big(\sum_{e \subset \tilde{\Gamma}} \|\nabla z_h\|^2_{L^2(e)}\Big)^{1/2}.\end{aligned}$$

Again using the fact that ∇v_h and ∇z_h are constant, we use this fact to write,

$$\|\nabla v_h\|^2_{L^2(e)} = |e| \big|(\nabla v_h)|_e\big|^2 = |\tau_e| \big|(\nabla v_h)|_{\tau_e}\big|^2 \frac{|e|}{|\tau_e|} = \frac{|e|}{|\tau_e|} |v_h|^2_{H^1(\tau_e)}, \tag{8}$$

where τ_e is the triangle assosciated with the boundary edge e. We use this identity, to write,

$$\begin{aligned}|a(v_h, z_h) - a_h(v_h, z_h)| &\leq Ch^{2+1/2} \frac{|e|}{|\tau_e|} \Big(\sum_{e \subset \tilde{\Gamma}} |v_h|^2_{H^1(\tau_e)}\Big)^{1/2} \Big(\sum_{e \subset \tilde{\Gamma}} |z_h|^2_{H^1(\tau_e)}\Big)^{1/2} \\ &\leq Ch^{1+1/2} \Big(\sum_{\tau \in \mathcal{T}_h} |v_h|^2_{H^1(\tau)}\Big)^{1/2} \Big(\sum_{\tau \in \mathcal{T}_h} |z_h|^2_{H^1(\tau)}\Big)^{1/2} \\ &= Ch^{1+1/2} |v_h|_{H^1(\Omega)} |z_h|_{H^1(\Omega)}.\end{aligned}$$

Now, notice from the definition of our operator norm, we have,

$$\begin{aligned}
\|a(v_h,\cdot) - a_h(v_h,\cdot)\|_{*,h} &= \sup_{z_h \in V_h} \frac{|a(v_h,z_h) - a_h(v_h,z_h)|}{\|z_h\|_{H^1(\Omega)}} \\
&\le \sup_{z_h \in V_h} \frac{Ch^{1+1/2}|v_h|_{H^1(\Omega)}|z_h|_{H^1(\Omega)}}{\|z_h\|_{H^1(\Omega)}} \\
&\le Ch^{1+1/2}|v_h|_{H^1(\Omega)}.
\end{aligned}$$

Looking back at the Bramble Hilbert lemma, we conclude,

$$\begin{aligned}
\inf_{v_h \in V_h} \left(\|u - v_h\|_{H^1(\Omega)} + \|a(v_h,\cdot) - a_h(v_h,\cdot)\|_{*,h}\right) &\le \inf_{v_h \in V_h} \left(\|u - v_h\|_{H^1(\Omega)} + Ch^{1+1/2}|v_h|_{H^1(\Omega)}\right) \\
&\le \|u - \Pi_h u\|_{H^1(\Omega)} + Ch^{1+1/2}\|\Pi_h u\|_{H^1(\Omega)} \\
&\le Ch|u|_{H^1(\Omega)} + Ch^{1+1/2}\|\Pi_h\|\|u\|_{H^1(\Omega)} \\
&\le Ch\|u\|_{H^1(\Omega)}.
\end{aligned}$$

where we have used the usual approximation properties on for $\|u - \Pi_h u\|_{H^1(\Omega)}$.

Now for $\|L - L_h\|_{*,h}$. Consider,

$$\begin{aligned}
\|L - L_h\|_{*,h} &= \sup_{v_h \in V_h} \frac{|L(v_h) - L_h(v_h)|}{\|v_h\|_{H^1(\Omega)}} \\
&= \sup_{v_h \in V_h} \frac{|\int_{\tilde{\Gamma}} v_h g \, ds - \mathcal{Q}_{\tilde{\Gamma}}(v_h, g)|}{\|v_h\|_{H^1(\Omega)}}.
\end{aligned}$$

We will focus on the numerator in the supremum,

$$\begin{aligned}
\Big| \int_{\tilde{\Gamma}} v_h g \, ds - \sum_{e \subset \tilde{\Gamma}} \frac{|e|}{2}((v_h g)(P_1^e) + (v_h g)(P_2^e))\Big| &\le \sum_{e \subset \tilde{\Gamma}} \Big| \int_e v_h g \, ds - \frac{|e|}{2}((v_h g)(P_1^e) + (v_h g)(P_2^e))\Big| \\
&\le \sum_{e \subset \tilde{\Gamma}} \frac{|e|}{2} \Big| \int_{-1}^{1} \hat{v}_h \hat{g} \, d\hat{s} - ((\hat{v}_h \hat{g})(-1) + (\hat{v}_h \hat{g})(1))\Big|.
\end{aligned}$$

Notice that we can apply the Bramble-Hilbert lemma in the same way as we did for the error in the bilinear forms, $a - a_h$. Therefore,

$$|L(v_h) - L_h(v_h)| \le C \sum_{e \subset \tilde{\Gamma}} \frac{|e|}{2} |\hat{v}_h \hat{g}|_{H^2(-1,1)} \le Ch^{2+1/2} \sum_{e \subset \tilde{\Gamma}} |v_h g|_{H^2(e)}.$$

Now consider,

$$\begin{aligned}
|v_h g|^2_{H^2(e)} &= \int_e \Big| \frac{d^2}{dx}(v_h g)\Big|^2 \, dx \\
&= \int_e |v_h'' g + 2v_h' g' + v_h g''|^2 \, dx \\
&\le C \int_e |v_h'|^2 |g'|^2 + |v_h|^2 |g''|^2 \, dx \\
&\le C\|g\|^2_{W^{2,\infty}(e)} \|v_h\|^2_{H^1(e)}.
\end{aligned}$$

Taking the square root and applying this inequality in our problem, we have,

$$
\begin{aligned}
|L(v_h) - L_h(v_h)| &\leq Ch^{2+1/2} \sum_{e \subset \tilde{\Gamma}} \|g\|_{W^{2,\infty}(e)} \|v_h\|_{H^1(e)} \\
&\leq Ch^{2+1/2} \|g\|_{W^{2,\infty}(\partial\Omega)} \sum_{e \subset \partial\Omega} \|v_h\|_{H^1(e)} \\
&\leq Ch^{2+1/2} \|g\|_{W^{2,\infty}(\partial\Omega)} \Big(\sum_{e \subset \partial\Omega} 1 \Big)^{1/2} \Big(\sum_{e \subset \partial\Omega} \|v_h\|^2_{H^1(e)} \Big)^{1/2} \\
&\leq Ch^2 \|g\|_{W^{2,\infty}(\partial\Omega)} \|v_h\|_{H^1(\partial\Omega)},
\end{aligned}
$$

where we have used the fact that the number of edges in our triangulation is proportional to $|\partial\Omega|/h$. Next note that, $\|v_h\|^2_{H^1(\partial\Omega)} = \|v_h\|^2_{L^2(\partial\Omega)} + \|\nabla v_h\|^2_{L^2(\partial\Omega)}$. Consider,

$$
\begin{aligned}
\|\nabla v_h\|^2_{L^2(\Omega)} &= \sum_{e \subset \partial\Omega} |v_h|^2_{H^1(e)} \\
&= \sum_{e \subset \partial\Omega} |e| \big| (\nabla v_h)|_e \big|^2 \\
&= \sum_{e \subset \partial\Omega} |\tau_e| \big| (\nabla v_h)|_{\tau_e} \big|^2 \frac{|e|}{|\tau_e|} \\
&\leq \frac{C}{h} \|\nabla v_h\|^2_{H^1(\Omega)}
\end{aligned}
$$

From this we can conclude that $\|v_h\|_{H^1(\partial\Omega)} \leq \frac{C}{h} \|v_h\|_{H^1(\Omega)}$. Therefore,

$$
|L(v_h) - L_h(v_h)| \leq Ch^{1+1/2} \|g\|_{W^{2,\infty}(\partial\Omega)} \|v_h\|_{H^1(\Omega)}. \tag{9}
$$

Therefore,

$$
\|L - L_h\|_{*,h} = \sup_{z_h \in V_h} \frac{|L(z_h) - L_h(z_h)|}{\|z_h\|_{H^1(\Omega)}} \leq Ch^{1+1/2} \|g\|_{W^{2,\infty}(\partial\Omega)}.
$$

Combining all of these results in Strang's lemma, we have,

$$
\|u - u_h\|_{H^1(\Omega)} \leq Ch\|u\|_{H^1(\Omega)} + Ch^{1+1/2} \|g\|_{W^{2,\infty}(\partial\Omega)} \leq Ch(\|u\|_{H^1(\Omega)} + \|g\|_{W^{2,\infty}(\partial\Omega)}).
$$

■

Problem 2. Consider the following initial boundary value problem: find $u(x,t)$ such that

$$\frac{\partial}{\partial t}(u - \Delta u) - \mu \Delta u = f, \; x \in \Omega, T \geq t > 0,$$
$$u(x,t) = 0, \; x \in \partial\Omega, \; T \geq t > 0,$$
$$u(x,0) = u_0(x), \; x \in \Omega,$$

where Ω is a polygonal domain in $\mathbb{R}^2$, $\mu > 0$ is a given constant, and $f(x,t)$ and $u_0(x)$ are given right hand side and initial data functions.

a. Derive a weak formulation of this problem and derive an *a priori* estimate for the solution in the norm

$$\|u(t)\|_{H^1(\Omega)} = (\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)})^{1/2}$$

in terms of the right-hand side and the initial data.

Solution: Multiplying the PDE by a test function $v(x)$, we have,

$$\int_\Omega (u_t - \Delta u_t)v - \mu \Delta u v \, dx = \int_\Omega u_t v + \nabla u_t \cdot \nabla v + \mu \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} (\frac{\partial u_t}{\partial n} + \mu \frac{\partial u}{\partial n}) v \, ds.$$

Since we have no information about $\partial u / \partial n$ on the boundary, we must take $v \in H^1_0(\Omega)$. Thus, our weak formulation becomes: find $u \in L^2(0,T; H^1_0(\Omega))$ such that

$$(u_t, v) + a(u_t, v) + \mu a(u,v) = (f, v),$$

for all $v \in H^1_0(\Omega)$, where $(\cdot,\cdot)$ is the usual L^2 inner product and $a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx$. Note that $a(\cdot,\cdot)$ is coercive, since our variational space is $H^1_0(\Omega)$ wich will imply that a Poincarè inequality exists. One can then prove that there exists $\alpha > 0$ such that $a(u,u) \geq \alpha \|u\|^2_{H^1(\Omega)}$ (coercivity).

Now, to find the a priori error estimate, set $v = u$ in the weak formulation. This gives us,

$$(u_t, u) + a(u_t, u) + \mu a(u,u) = \int_\Omega \frac{1}{2}\frac{d}{dt}|u|^2 + \frac{1}{2}\frac{d}{dt}|\nabla u|^2 \, dx + \mu a(u,u)$$
$$= \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^1(\Omega)} + \mu a(u,u)$$

Applying the usual inequalities, we can write

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^1(\Omega)} + \alpha\mu\|u(t)\|^2_{H^1(\Omega)} \leq (f,u) \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}.$$

This can be rewritten as,

$$\frac{d}{dt}\|u(t)\|_{H^1(\Omega)} + \alpha\mu\|u(t)\|_{H^1(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$

Multiplying both sides of the inequality by $e^{\alpha\mu t}$ and using the product rule (integrating factors), we have,

$$\frac{d}{dt}\big(e^{\alpha\mu t}\|u(t)\|_{H^1(\Omega)}\big) \le e^{\alpha\mu t}\|f\|_{L^2(\Omega)}.$$

Now replace t by s as the dummy variable and integrate from 0 to t. This gives us,

$$e^{\alpha\mu t}\|u(t)\|_{H^1(\Omega)} - \|u(0)\|_{H^1(\Omega)} \le \int_0^t e^{\alpha\mu s}\|f(s)\|_{L^2(\Omega)}\,ds.$$

Solving for $\|u(t)\|_{H^1(\Omega)}$ gives us the a priori estimate,

$$\|u(t)\|_{H^1(\Omega)} \le e^{-\alpha\mu t}\|u_0\|_{H^1(\Omega)} + \int_0^t e^{-\alpha\mu(t-s)}\|f(s)\|_{L^2(\Omega)}\,ds.$$

■

b. Write down the fully discrete scheme based on implicit (backward) Euler approximation in time and the finite element method in space with continuous piece-wise linear functions. **Prove** unconditional stability in the H^1-norm for the resulting approximation.

Solution: Let V_h be the space of continuous piece-wise linear functions and let the basis (tent) functions be denoted as $V_h = \text{span}\{\phi_i\}_{i=1}^N$. Additionally, let $t_n = nk$ for $0 \le n \le N$ an integer and $k > 0$ such that $T = kN$, then define $u_h^n := u_h(x, t_n)$ and $f^n := f(x, t_n)$. Then our fully discrete problem becomes: given $u_h^n \in V_h$, find $u_h^{n+1} \in V_h$ such that,

$$\Big(\frac{u_h^{n+1} - u_h^n}{k}, v_h\Big) + a\Big(\frac{u_h^{n+1} - u_h^n}{k}, v_h\Big) + \mu a(u_h^{n+1}, v_h) = (f^{n+1}, v_h),$$

for all $v_h \in V_h$ and $u_h^0 = \Pi_h u_0$, that is, the projection of u_0 onto the space V_h.

Now to prove stability for this approximation, set $v_h = u_h^{n+1}$, and rewrite the terms; we then have,

$$\begin{aligned}\|u_h^{n+1}\|^2_{L^2(\Omega)} - (u_h^n, u_h^{n+1}) + \|\nabla u_h^{n+1}\|^2_{L^2(\Omega)} - a(u_h^n, u_h^{n+1}) + k\mu a(u_h^{n+1}, u_h^{n+1})& \\ \le k\|f^{n+1}\|_{L^2(\Omega)}\|u_h^{n+1}\|_{L^2(\Omega)}&.\end{aligned}$$

Rearranging the terms and dropping $k\mu a(u_h^{n+1}, u_h^{n+1})$, we get,

$$\|u_h^{n+1}\|^2_{H^1(\Omega)} \le k\|f^{n+1}\|_{L^2(\Omega)}\|u_h^{n+1}\|_{L^2(\Omega)} + \|u_h^n\|_{L^2(\Omega)}\|u_h^{n+1}\|_{L^2(\Omega)} + |u_h^n|_{H^1(\Omega)}|u_h^{n+1}|_{H^1(\Omega)}$$

Next, using the inequality, $ab + cd \le \sqrt{a^2 + c^2}\sqrt{b^2 + d^2}$, we have,

$$\|u_h^{n+1}\|^2_{H^1(\Omega)} \le k\|f^{n+1}\|_{L^2(\Omega)}\|u_h^{n+1}\|_{H^1(\Omega)} + \|u_h^n\|_{H^1(\Omega)}\|u_h^{n+1}\|_{H^1(\Omega)}.$$

We divide by $\|u_h^{n+1}\|_{H^1(\Omega)}$ which gives us,

$$\|u_h^{n+1}\|_{H^1(\Omega)} \leq k\|f^{n+1}\|_{L^2(\Omega)} + \|u_h^n\|_{H^1(\Omega)} \leq \cdots \leq k \sum_{j=1}^{n+1} \|f^j\|_{L^2(\Omega)} + \|u_h^0\|_{H^1(\Omega)} < \infty.$$

We assume that $f \in L^1(0,T;L^2(\Omega))$, which allows us to guarentee the right hand side is finite. Thus we have unconditional stability in the H^1-norm. (Unconditional in the sense that, the stability is not dependent on the spacial or temporal mesh sizes.) ■

c. Consider now the forward Euler approximation for the derivative in t. **Find** the Courant condition for stability of the resulting method in a norm of your choice.

Solution: Couldn't figure this one out.

Problem 3. Let $\mathcal{T}_h$ be a partition of $(0,1)$ into finite elements of equal size $h = 1/N$, $N > 1$ an integer, and $x_i = ih$, $i = 0, 1, \dots, N$. Consider the finite dimensional space V_h of continuous piece-wise quadratic functions on $\mathcal{T}_h$. The degrees of freedom on finite element (x_{i-1}, x_i) are

$$\Big\{ v(x_{i-1}), v(x_i), \frac{1}{h}\int_{x_{i-1}}^{x_i} v\,dx \Big\}. \tag{10}$$

a. Explicitly find the nodal basis of V_h over the finite element (x_{i-1}, x_i), corresponding to these degrees of freedom.

Solution: Let ϕ_1, ϕ_2, and ϕ_3 be the nodal basis functions for V_h over (x_{i-1}, x_i). So to find ϕ_1, it must satisfy the following: $\phi_1(x_{i-1}) = 1$, $\phi_1(x_i) = 0$ and $\frac{1}{h}\int_{x_{i-1}}^{x_i} \phi_1\,dx = 0$. To make things simpler, we first transfer to the reference element to determine ϕ_1 and then transfer back. So using the map $T_i : [0,1] \to [x_{i-1}, x_i]$ defined by $x = T_i(\hat{x}) = h\hat{x} + x_{i-1}$. We define $\hat{\phi}_{1,i} := \phi_1 \circ T_i$, and we will drop the i subscript to alleviate the notation. Then we have,

$$\hat{\phi}_1(0) = 1, \quad \hat{\phi}_1(1) = 0, \quad \text{and} \quad \int_0^1 \hat{\phi}_1(\hat{x})\,d\hat{x} = 0.$$

Thus for $\hat{\phi}_1 = a\hat{x}^2 + b\hat{x} + c$, we have,

$$\begin{aligned}
\hat{\phi}_1(0) &= c = 1 \\
\hat{\phi}_1(1) &= a + b + c = 0 \\
\int_0^1 \hat{\phi}_1(\hat{x})\,d\hat{x} &= \frac{1}{3}a + \frac{1}{2}b + c = 0.
\end{aligned}$$

Solving this system of equations tells us that,

$$\hat{\phi}_1(x) = 3\hat{x}^2 - 4\hat{x} + 1.$$

We transfer back to the global element and repeat this for the other functions ϕ_2 and ϕ_3. We conclude that,

$$\begin{aligned}
\phi_1(x) &= 3\Big(\frac{x - x_{i-1}}{h}\Big)^2 - 4\Big(\frac{x - x_{i-1}}{h}\Big) + 1 \\
\phi_2(x) &= 3\Big(\frac{x - x_{i-1}}{h}\Big)^2 - 2\Big(\frac{x - x_{i-1}}{h}\Big) \\
\phi_3(x) &= -6\Big(\frac{x - x_{i-1}}{h}\Big)^2 + 6\Big(\frac{x - x_{i-1}}{h}\Big).
\end{aligned}$$

■

b. Prove that sup

$$\sup_{\phi \in H_0^1(\Omega)} \frac{\displaystyle\int_0^1 (u - \Pi_h u)\phi \, dx}{||\phi||_{H_0^1(\Omega)}} \leq Ch||u - \Pi_h u||_{L^2(0,1)}, \tag{11}$$

$\forall u \in H^1(0,1)$. Here $\Pi_h u$ is the finite element interpolant of u with respect to the nodal basis of V_h defined by (10).

Proof: First, we will show that $\int_0^1 u - \Pi_h u \, dx = 0$. So consider,

$$\begin{aligned}
\int_0^1 u - \Pi_h u \, dx &= \sum_{i=1}^N \int_{x_{i-1}}^{x_i} u - \Pi_h u \, dx \\
&= \sum_{i=1}^N \int_{x_{i-1}}^{x_i} u(x) - u(x_{i-1})\phi_{1,i}(x) - u(x_i)\phi_{2,i}(x) - \Big(\frac{1}{h}\int_{x_{i-1}}^{x_i} u(s)\, ds\Big)\phi_{3,i}(x)\, dx \\
&= \sum_{i=1}^N \Big\{ \int_{x_{i-1}}^{x_i} u(x)\, dx - \Big(\int_{x_{i-1}}^{x_i} u(s)\, ds\Big)\Big(\frac{1}{h}\int_{x_{i-1}}^{x_i} \phi_{3,i}(x)\, dx\Big)\Big\} \\
&= 0,
\end{aligned}$$

where we have used the fact that the nodal basis functions satisfy $\sigma_{k,i}(\phi_{j,i}) = \delta_{kj}$ for the linear functionals $\sigma_{k,i}$ defined in part a. (the sigma notation is not used in part a.). Thus, we can say that $\int_0^1 c(u - \Pi_h u)dx = 0$ for any constant c. In particular, $\int_{x_{i-1}}^{x_i} c(u - \Pi_h u)dx = 0$ for any constant c. Now define $\bar{\phi}(x)$ to be a piecewise constant function, such that $\bar{\phi}(x) = \bar{\phi}_i := \frac{1}{h}\int_{x_{i-1}}^{x_i} \phi(x)\, dx$ for $x \in (x_{i-1}, x_i)$, $i = 1, \ldots, N$. Then we can write,

$$\begin{aligned}
\int_0^1 (u - \Pi_h u)\phi \, dx &= \sum_{i=1}^N \int_{x_{i-1}}^{x_i} (u - \Pi_h u)\phi \, dx \\
&= \sum_{i=1}^N \int_{x_{i-1}}^{x_i} (u - \Pi_h u)(\phi - \bar{\phi}_i)\, dx \\
&\leq \sum_{i=1}^N ||u - \Pi_h u||_{L^2(x_{i-1},x_i)} ||\phi - \bar{\phi}_i||_{L^2(x_{i-1},x_i)}.
\end{aligned}$$

Transfering to the reference element with the mapping T_i as in part a., we can write,

$$\begin{aligned}
||\phi - \bar{\phi}_i||^2_{L^2(x_{i-1},x_i)} &= \int_{x_{i-1}}^{x_i} (\phi(x) - \frac{1}{h}\int_{x_{i-1}}^{x_i} \phi(s)\, ds)^2 \, dx \\
&= \int_0^1 (\hat{\phi} - \frac{1}{h}\int_{x_{i-1}}^{x_i} \phi(s)\, ds)^2 h \, d\hat{x} \\
&= h\int_0^1 (\hat{\phi} - \int_0^1 \hat{\phi}\, d\hat{s})^2 \, d\hat{x} \\
&= h||\hat{\phi} - \bar{\hat{\phi}}_i||^2_{L^2(0,1)}
\end{aligned}$$

So note that $L(\hat{\phi}) := \|\hat{\phi} - \bar{\hat{\phi}}_i\|_{L^2(0,1)}$ is a sublinear functional on $H^1(0,1)$ which is zero for constants. By the Bramble-Hilbert lemma, we have that $\|\hat{\phi} - \bar{\phi}\|_{L^2(0,1)} \le C|\hat{\phi}|_{H^1(0,1)}$. Applying these results and transferring back to the element $[x_{i-1}, x_i]$, we have,

$$\begin{aligned}
\int_0^1 (u - \Pi_h u)\phi \, dx &\le Ch^{1/2} \sum_{i=1}^N \|u - \Pi_h u\|_{L^2(x_{i-1},x_i)} |\hat{\phi}|_{H^1(0,1)} \\
&\le Ch^{1/2} \sum_{i=1}^N \|u - \Pi_h u\|_{L^2(x_{i-1},x_i)} \cdot Ch^{1/2} |\phi|_{H^1(x_{i-1},x_i)} \\
&\le Ch \|u - \Pi_h u\|_{L^2(0,1)} \|\phi\|_{H^1(0,1)}.
\end{aligned}$$

Therefore,

$$\sup_{\phi \in H_0^1(\Omega)} \frac{\int_0^1 (u - \Pi_h u)\phi \, dx}{\|\phi\|_{H^1(0,1)}} \le Ch \|u - \Pi_h u\|_{L^2(0,1)}.$$

■