Applied/Numerical Qualifier Solution: January 2010

Bennett Clayton

Texas A&M University

July 19, 2021

Problem 4. Let $\mathcal{H}$ be a complex (separable) Hilbert space, with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ being the inner product and norm.

a.

Solution:

b. Briefly explain why the operator $Ku(x) := \int_0^1 (3+4xy^2)u(y)\,dy$ is compact on $\mathcal{H} = L^2[0,1]$. Determine the values of $\lambda \in \mathbb{C}$ for which $u = f + \lambda K u$ has a solution for all $f \in L^2[0,1]$. State the theorem that you are using to answer the question.

Solution: Note that K is a Hilbert-Schmidt operator and hence is compact. First define $L := I - \lambda K$, then we want to determine $N(L^*) = N(I - \bar{\lambda}K^*)$. Then since K is Hilbert-Schmidt, K^* will be

$$(K^*u)(x) = \int_0^1 (3 + 4yx^2)u(y)\,dy$$

Lets now set $L^*u = 0$, and take some observations

$$\begin{aligned} L^*u &= u(x) - \bar{\lambda}\int_0^1 (3 + 4yx^2)u(y)\,dy = 0 \\ &= u(x) - 3\bar{\lambda}\int_0^1 u(y)\,dy - 4\bar{\lambda}x^2 \int_0^1 yu(y)\,dy = 0 \end{aligned}$$

We now consider two different equations. First we integrate the equation with respect to x and integrate the equation against x, again with respect to x. Lets start with finding the first relation

$$\int_0^1 u(x)\,dx - 3\bar{\lambda}\int_0^1 u(y)\,dy - \int_0^1 4\bar{\lambda}x^2\int_0^1 yu(y)\,dy\,dx = 0$$
$$(1-3\bar{\lambda})\int_0^1 u(y)\,dy - \frac{4}{3}\bar{\lambda}\int_0^1 yu(y)\,dy = 0$$

Setting $I_1 = \int_0^1 u(y)\,dy$ and $I_2 = \int_0^1 yu(y)\,dy$ we have the following relation

$$(1-3\bar{\lambda})I_1 - \frac{4}{3}\bar{\lambda}I_2 = 0$$

Now for the next relation, multiply by x and integrate the original equation, i.e. we have

$$\int_0^1 xu(x)\,dx - \int_0^1 x3\bar{\lambda}\int_0^1 u(y)\,dy\,dx - \int_0^1 4\bar{\lambda}x^3\int_0^1 yu(y)\,dy\,dx = 0$$
$$(1-\bar{\lambda})\int_0^1 xu(x)\,dx - \frac{3}{2}\bar{\lambda}\int_0^1 u(y)\,dy = 0$$

Thus our second equation is

$$-\frac{3}{2}\bar{\lambda}I_1 + (1-\bar{\lambda})I_2 = 0$$

Proof:

Problem 1. Consider the system

$$\begin{aligned} -\Delta u - \phi &= f \\ u - \Delta\phi &= g \end{aligned} \tag{1}$$

in the bounded smooth domain Ω, with boundary conditions $u = \phi = 0$ on $\partial\Omega$.

a. Derive a weak formulation of the system (1), using suitable test functions for each equation. Define a bilinear form $a((u,\phi),(v,\psi))$ such that this weak formulation amounts to,

$$a((u,\phi),(v,\psi)) = (f,v) + (g,\psi). \tag{2}$$

Proof: Multiplying the first equation by a test function v and the second equation by a test function ψ, we then integrate. Doing so gives us,

$$\int_{\Omega} -\Delta uv - \phi v \, d\boldsymbol{x} = -\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds + \int_{\Omega} \nabla u \cdot \nabla v - \phi v \, d\boldsymbol{x}$$
$$\int_{\Omega} u\psi - \Delta\phi\psi \, d\boldsymbol{x} = -\int_{\partial\Omega} \frac{\partial \phi}{\partial n} \psi \, ds + \int_{\Omega} u\psi + \nabla\phi \cdot \nabla\psi \, d\boldsymbol{x}.$$

Adding the two equations together and taking the essential boundary condition that $(v, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$, we have,

$$a((u,\phi),(v,\psi)) := \int_{\Omega} \nabla u \cdot \nabla v + \nabla\phi \cdot \nabla\psi + u\psi - \phi v \, d\boldsymbol{x} = (f, v) + (g, \psi). \tag{3}$$

b. Choose appropriate function spaces for u and ϕ in a.

Proof: We simply take $(u, \phi) \in V := H_0^1(\Omega) \times H_0^1(\Omega)$. So our ansatz and test function spaces coincide.

c. Show that the weak formulation in a. has a unique solution. Hint: Lax-Milgram.

Proof: We want to apply Lax-Milgram to guarantee a unique solution, but we first need to prove continuity and coercivity of our bilinear forms. We will use the norm,

$$|||(u,\phi)||| = \sqrt{||u||^2_{H^1(\Omega)} + ||\phi||^2_{H^1(\Omega)}} \tag{4}$$

to prove the results. Similarly, since u and ϕ are both in $H_0^1(\Omega)$ we will have a Poincarè inequality, therefore, we have,

$$\begin{aligned}
a((u,\phi),(u,\phi)) &= \int_\Omega |\nabla u|^2 + |\nabla \phi|^2 + u\phi - \phi u \, dx \\
&= \int_\Omega |\nabla u|^2 + |\nabla \phi|^2 \, dx \\
&= |u|^2_{H^1(\Omega)} + |\phi|^2_{H^1(\Omega)} \\
&= \frac{1}{2}|u|^2_{H^1(\Omega)} + \frac{1}{2}|\phi|^2_{H^1(\Omega)} + \frac{1}{2}|u|^2_{H^1(\Omega)} + \frac{1}{2}|\phi|^2_{H^1(\Omega)} \\
&\geq \frac{1}{2}|u|^2_{H^1(\Omega)} + \frac{1}{2}|\phi|^2_{H^1(\Omega)} + \frac{C}{2}\|u\|^2_{L^2(\Omega)} + \frac{C}{2}\|\phi\|^2_{L^2(\Omega)} \\
&\geq \frac{1}{2}\min\{1, C\}\big(\|u\|^2_{H^1} + \|\phi\|^2_{H^1}\big) \\
&= \alpha |||(u,\phi)|||^2.
\end{aligned}$$

Thus a is coercive. Next we will show a is continuous,

$$\begin{aligned}
a((u,\phi),(v,\psi)) &= \int_\Omega \nabla u \cdot \nabla v + \nabla \phi \cdot \nabla \psi + u\psi - \phi v \, dx \\
&\leq |u|_{H^1(\Omega)}|v|_{H^1(\Omega)} + |\phi|_{H^1(\Omega)}|\psi|_{H^1(\Omega)} + \|u\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} + \|\phi\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\leq \|u\|_{H^1(\Omega)}(|v|_{H^1(\Omega)} + \|\psi\|_{L^2(\Omega)}) + \|\phi\|_{H^1(\Omega)}(|\psi|_{H^1(\Omega)} + \|v\|_{L^2(\Omega)}) \\
&\leq (\|u\|_{H^1(\Omega)} + \|\phi\|_{H^1(\Omega)})(\|v\|_{H^1(\Omega)} + \|\psi\|_{H^1(\Omega)}) \\
&\leq \sqrt{2}|||(u,\phi)||| \cdot \sqrt{2}|||(v,\psi)||| \\
&= 2|||(u,\phi)||| \cdot |||(v,\psi)|||.
\end{aligned}$$

It is also easy to check that the right hand side $(f,v) + (g,\psi)$ is continuous. Therefore by Lax-Milgram, there exists a unique solution.

d. For a domain $\Omega_d = (-d,d)^2$, show that

$$\|u\|^2 \leq cd^2\|\nabla u\|^2 \tag{5}$$

holds for any function $u \in H^1_0(\Omega_d)$.

Proof: Consider,

$$
\begin{aligned}
\|u\|^2 &= \int_{-d}^{d}\int_{-d}^{d} u(x,y)^2 \, dx \, dy \\
&= \int_{-d}^{d}\int_{-d}^{d} \frac{1}{2}u(x,y)^2 + \frac{1}{2}u(x,y)^2 \, dx \, dy \\
&= \int_{[-d,d]^2} \frac{1}{2}\Big(\int_{-d}^{x} \frac{\partial}{\partial \xi}u(\xi,y)\, d\xi\Big)^2 + \frac{1}{2}\Big(\int_{-d}^{y} \frac{\partial}{\partial \eta}u(x,\eta)\, d\eta\Big)^2 \, dx \, dy \\
&\le \frac{1}{2}\int_{[-d,d]^2} \Big[(x+d)\int_{-d}^{x}\Big(\frac{\partial}{\partial \xi}u(\xi,y)\Big)^2 d\xi + (y+d)\int_{-d}^{y}\Big(\frac{\partial}{\partial \eta}u(x,\eta)\Big)^2 d\eta\Big] \, dx \, dy \\
&\le d\int_{[-d,d]}\int_{-d}^{d}\Big(\frac{\partial}{\partial \xi}u(\xi,y)\Big)^2 d\xi + \int_{-d}^{d}\Big(\frac{\partial}{\partial \eta}u(x,\eta)\Big)^2 d\eta \, dx \, dy \\
&= 2d^2 \int_{\Omega_d} |\nabla u|^2 \, dx \, dy \\
&= 2d^2 \|\nabla u\|^2.
\end{aligned}
$$

■

e. Now change the second "−" in the first equation of (1) to a "+". Use d. to show stability for the modified equation on Ω_d, provided that d is sufficiently small.

Proof: If we repeat the same process for this new set of equations, we will end up with the bilinear form,

$$
a((u,\phi),(v,\psi)) = \int_{\Omega_d} \nabla u \cdot \nabla v + \nabla \phi \cdot \nabla \psi + u\psi + \phi v \, dx
$$

Now take d small enough such that $1/(2cd^2) \ge 1$. Then, we have,

$$
\begin{aligned}
a((u,\phi),(u,\phi)) &= \int_{\Omega_d} |\nabla u|^2 + |\nabla \phi|^2 + 2u\phi \, dx \, dy \\
&= \int_{\Omega_d} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla \phi|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla \phi|^2 + 2u\phi \, dx \, dy \\
&\ge \int_{\Omega_d} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla \phi|^2 + \frac{1}{4d^2}(|u|^2 + |\phi|^2) + 2u\phi \, dx \, dy.
\end{aligned}
$$

Take d small enough so that $\frac{1}{4d^2} \geq 1$. Using this inequality, we have,

$$\begin{aligned}
a((u,\phi),(u,\phi)) &\geq \int_{\Omega_d} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla \phi|^2 + u^2 + \phi^2 + 2u\phi \, dx \, dy \\
&= \int_{\Omega_d} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla \phi|^2 + (u+\phi)^2 \, dx \, dy \\
&\geq \frac{1}{2}(|u|^2_{H^1(\Omega_d)} + |\phi|^2_{H^1(\Omega_d)}) \\
&\geq \frac{1}{4}\Big(|u|^2_{H^1(\Omega_d)} + |\phi|^2_{H^1(\Omega_d)} + \frac{1}{2d^2}(||u||^2_{L^2(\Omega_d)} + ||\phi||^2_{L^2(\Omega_d)})\Big) \\
&\geq \frac{1}{2}(||u||^2_{H^1(\Omega_d)} + ||\phi||^2_{H^1(\Omega_d)}) \\
&= \frac{1}{2}|||(u,\phi)|||^2.
\end{aligned}$$

Thus a is coercive. How does this relate to stability?

Problem 2. Consider the two finite elements (τ, Q_1, Σ) and $(\tau, \tilde{Q}_1, \Sigma)$ where $\tau = [-1,1]^2$ is the reference square and

$$Q_1 = \text{span}\{1, x, y, xy\} \tag{6}$$
$$\tilde{Q}_1 = \text{span}\{1, x, y, x^2 - y^2\}. \tag{7}$$

$\Sigma = \{w(1,0), w(-1,0), w(0,1), w(0,-1)\}$ is the set of values of a function $w(x,y)$ at the midpoints of the edges τ.

a. Which of the two elements is unisolvent? Prove it!

Proof: To show unisolvence, we need to show that if the linear functionals all evaluate to zero for an arbitrary function in Q_1 or $\tilde{Q}_1$ then our function must be identically zero. So let $w(x,y) = a + bx + cy + dxy \in Q_1$. Then we have

$$\begin{aligned}
w(1,0) &= a + b = 0 \\
w(-1,0) &= a - b = 0 \\
w(0,1) &= a + c = 0 \\
w(0,-1) &= a - c = 0.
\end{aligned}$$

solving these systems of equations, we can conclude that $a = b = c = 0$. However, d can be any value, say $d \neq 0$, then w is not identically zero, hence (τ, Q_1, Σ) is not unisolvent.

For the other finite element, we take $w(x,y) = a + bx + cy + d(x^2 - y^2) \in \tilde{Q}_1$. Performing the same process, we have,

$$\begin{aligned} w(1,0) &= a + b + d = 0 \\ w(-1,0) &= a - b + d = 0 \\ w(0,1) &= a + c - d = 0 \\ w(0,-1) &= a - c - d = 0. \end{aligned}$$

So if we solve this linear system, we find that $a = b = c = d = 0$. Thus $w(x,y) \equiv 0$ and therefore $(\tau, \tilde{Q}_1, \Sigma)$ is unisolvent.

b. Show that the unisolvent element leads to a finite element space, which is not H^1-conforming.

Proof: To show this, we only need to show a specific example. So let K_1 be the square $[-1,1] \times [-1,1]$ and K_2 be the square formed by the vertices $\{(1,-1),(3,-1),(3,1),(1,1)\}$. We define $\Omega := K_1 \cup K_2$. Then the unisolvent element leads to the following finite element space,

$$V_h := \{v : \Omega \to \mathbb{R} : v \circ T_{K_i} \in \tilde{Q}_1, \text{ for } i = 1,2, \text{ and } v|_{K_1}(1,0) = v|_{K_2}(1,0)\}, \tag{8}$$

where $T_{K_i} : \tau \to K_i$. We claim that $V_h \notin H^1(\Omega)$. So let $w \in V_h$ and define $w_1 := w|_{K_1}$ and $w_2 := w|_{K_2}$, where

$$\begin{aligned} w_1(x,y) &:= x^2 - y^2, \\ w_2(x,y) &:= x. \end{aligned}$$

Then notice that $w_1(1,0) = w_2(1,0) = 1$, but $w_1(1,y) = 1 - y^2 \neq 1 = w_2(1,y)$ for $y \neq 0$. Therefore $w \notin H^1(\Omega)$. (Note, the discontinuity is not removable.) Hence the finite element space is not be H^1-conforming. ■

Problem 3. Consider the following initial boundary value problem: find $u(x,t)$ such that

$$u_t - u_{xx} + u = 0, \qquad 0 < x < 1, \quad t > 0 \tag{9}$$

$$u_x(0,t) = u_x(1,t) = 0, \qquad t > 0 \tag{10}$$

$$u(x,0) = g(x), \quad 0 < x < 1. \tag{11}$$

a. Derive the semi-discrete approximation of this problem using linear finite elements over a uniform partition of $(0,1)$. Write it as a system of linear ordinary differential equations for the coefficient vector.

Proof: First, we mulitply the PDE by a test function $v(x)$ in some space V and integrate over $(0,1)$, doing so gives us,

$$\begin{aligned}0 &= \int_0^1 u_t(x,t)v(x) - u_{xx}(x,t)v(x) + u(x,t)v(x)\,dx\\ &= (u_t, v) + \int_0^1 u_x(x,t)v_x(x) + u(x,t)v(x)\,dx - u_x(x,t)v(x)\Big|_0^1\\ &= (u_t, v) + a(u,v),\end{aligned}$$

where

$$\begin{aligned}(u,v) &= \int_0^1 uv\,dx\\ a(u,v) &= \int_0^1 u_x v_x + uv\,dx.\end{aligned}$$

In this case our function space will be $H^1(0,1)$. Let $K_i := [x_{i-1}, x_i]$ and $\mathcal{T}_h := \{K_i\}_{i=1}^N$. Let $\widehat{K} := [0,1]$ be the reference shape and define the affine geometric mappings, $T_{K_i} : \widehat{K} \to K_i$, by $T_{K_i}(\hat{x}) = h\hat{x} + x_{i-1}$. Then our finite element space is defined as,

$$V_h := \{v \in C^0(0,1) : v \circ T_K \in \mathbb{P}_1, \forall K \in \mathcal{T}_h\}.$$

Our weak formulation in the finite element discretization is, find $u_h(t) \in V_h$ such that $\frac{d}{dt}(u_h(t), v_h) + a(u_h(t), v_h) = 0$ for all $v_h \in V_h$ with $u_h(0) = u_h^0(x)$ being an approximation of $g(x)$.

The basis for V_h consists of the usual nodal Lagrange shape functions (tent functions); that is, $V_h = \text{span}\{\phi_i\}_{i=0}^N$. Set the test function, $v = \phi_j(x)$, and express $u_h(t)$ in terms of the basis functions,

$$u_h(x,t) = \sum_{i=0}^N u_i(t)\phi_i(x).$$

Here, the $u_i(t)$ are the unknown coefficients. Additionally, our initial data is expressed as, $u_h^0(x) = \sum_{i=0}^N g_i\phi_i(x)$ (where g_i are known coefficients). Plugging this all into our variational equation, we find,

$$\int_0^1 \sum_{i=0}^N u_i'(t)\phi_i(x)\phi_j(x)\,dx + \int_0^1 \sum_{i=0}^N u_i(t)\phi_i'(x)\phi_j'(x) + \sum_{i=0}^N u_i(t)\phi_i(x)\phi_j(x)\,dx,$$

for $j = 0, \dots, N$. Let $U(t) := (u_0(t), \dots, u_N(t))^T$, M and A matrices with entries $m_{ij} = (\phi_i, \phi_j)$ and $a_{ij} = a(\phi_i, \phi_j)$, respectively. Then our system of equations is written in the following matrix form,

$$\begin{cases} MU'(t) + AU(t) = 0,\\ U(0) = G,\end{cases}$$

where $G = (g_0, \ldots, g_N)^T$. This is the system of linear ordinary differential equations. ■

b. Further, derive discretizations in time using backward Euler and Crank-Nicolson methods, respectively.

Proof: For the backward Euler we have,

$$(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h) + a(u_h^{n+1}, v_h) = 0 \tag{12}$$

where $u_h^n = u_h(x, t^n)$. For the Crank-Nicolson method,

$$(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h) + a(\frac{u_h^{n+1} + u_h^n}{2}, v_h) = 0 \tag{13}$$

These methods can also be written in matrix form. For the backward Euler we have,

$$M\frac{U^{n+1} - U^n}{\Delta t} + AU^{n+1} = 0, \tag{14}$$

where $U^n = U(n\Delta t)$. We can solve for U^{n+1} which gives,

$$U^{n+1} = (M + \Delta t A)^{-1} M U^n.$$

For the Crank-Nicolson method, we have,

$$M\frac{U^{n+1} - U^n}{\Delta t} + A\frac{U^{n+1} + U^n}{2} = 0.$$

Solving for U^{n+1}, we have,

$$U^{n+1} = (M + \frac{\Delta t}{2}A)^{-1}(M - \frac{\Delta t}{2}A)U^n.$$

■

c. Show that both fully discrete schemes are unconditionally stable with respect to the initial data in the spatial $L^2(0,1)$-norm.

Proof: To prove the stability, we use the variational form, so consider,

$$(\frac{u^{n+1} - u^n}{\Delta t}, v) + a(u^{n+1}, v) = 0,$$

where $u^n = u(x, n\Delta t)$. Then if we take $v = u^{n+1}$, we have,

$$(\frac{u^{n+1} - u^n}{\Delta t}, u^{n+1}) + a(u^{n+1}, u^{n+1}) = 0.$$

Note that $a(u^{n+1}, u^{n+1}) \geq 0$, so dropping that term, and rearranging, we end up with the inequality,

$$||u^{n+1}||^2_{L^2(0,1)} = (u^{n+1}, u^{n+1}) \leq (u^n, u^{n+1}) \leq ||u^n||_{L^2(0,1)} ||u^{n+1}||_{L^2(0,1)}.$$

Hence we have

$$||u^{n+1}||_{L^2(0,1)} \leq ||u^n||_{L^2(0,1)} \leq \cdots \leq ||u^0||_{L^2(0,1)} = ||g||_{L^2(0,1)}.$$

Thus the backward Euler is unconditionally stable.

Now for the Crank-Nicolson method,

$$(\frac{u^{n+1} - u^n}{\Delta t}, v) + a(\frac{u^{n+1} + u^n}{2}, v) = 0.$$

We repeat the same tricks as we did for the backward Euler, except this time we set $v = \frac{u^{n+1}+u^n}{2}$. This gives us,

$$(\frac{u^{n+1} - u^n}{\Delta t}, \frac{u^{n+1} + u^n}{2}) \leq 0.$$

If we expand the inner product and rearrange terms, we end up with the inequality,

$$||u^{n+1}||^2_{L^2(0,1)} \leq ||u^n||^2_{L^2(0,1)}.$$

Hence by the same arguments as before, we have,

$$||u^{n+1}||_{L^2(0,1)} \leq ||g||_{L^2(0,1)}.$$

Thus the Crank-Nicolson method is unconditionally stable. ■