Applied/Numerical Qualifier Solution: January 2011

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Problem 1. Consider the following two-points boundary value second order problem in 1-D: Find a function u define a.e. in (0,1) such that

$$-(xK(x)u'(x))' + xq(x)u(x) = xf(x) \text{ a.e. in } (0,1), \quad (1)$$

$$\lim_{x\to 0}(xu'(x)) = 0 \text{ and } K(1)u'(1) + u(1) = 0, \quad (2)$$

where $K \in \mathcal{C}^1([0,1])$ and $f \in L^2(0,1)$ are given functions. Assume that there exists a constant $\kappa_0 > 0$ such that $K(x) \geq \kappa_0$ and $q(x) \geq 0$ for all $x \in [0,1]$. Let

$$V = \{v \in L^2_{\text{loc}}(0,1) : \sqrt{x}v \in L^2(0,1), \sqrt{x}v' \in L^2(0,1)\}$$

Accept as a fact that V is a Hilbert space for the norm

$$\|v\|_V = \left(\|\sqrt{x}v\|^2_{L^2(0,1)} + \|\sqrt{x}v'\|^2_{L^2(0,1)}\right)^{1/2}$$

and $\mathcal{C}^1([0,1])$ is dense in V for this norm.

a. Derive the variational formulation (also called weak formulation) of problem 1 in the space V.

Solution: Taking $v \in V$, multiply the equation (1) and integrate to get the variational form.

$$\begin{aligned}
\int_0^1 -(xKu')'v + xquvdx &= \int_0^1 xKu'v' + xquvdx - \lim_{t\to 0}\left[xKu'v\right]_t^1 \\
&= \int_0^1 xKu'v' + xquvdx - K(1)u'(1)v(1) + \lim_{t\to 0} tK(t)u'(t)v(t) \\
&= \int_0^1 xKu'v' + xquvdx + u(1)v(1) + (\lim_{t\to 0} tu'(t))(K(0)v(0)) \\
&= \int_0^1 xKu'v' + xquvdx + u(1)v(1)
\end{aligned}$$

Thus we have $a(u,v) := \int_0^1 xKu'v' + xquvdx + u(1)v(1)$ and $F(v) := \int_0^1 xfvdx$. ■

b. Prove that the corresponding bilinear form of this variational formulation is elliptic (or coercive) in V. **Hint.** First show that all functions v of $C^1([0,1])$ satisfy

$$\int_0^1 v(x)^2\,dx = v^2(1) - 2\int_0^1 xv(x)v'(x)\,dx \tag{3}$$

and then establish the following variant Poincaré's inequality

$$\forall v \in V, \quad \|\sqrt{x}v\|_{L^2(0,1)} \le \alpha\left(v^2(1) + \|\sqrt{x}v'\|^2_{L^2(0,1)}\right)^{1/2} \tag{4}$$

for some constant $\alpha > 0$. Based on this equality deduct the ellipticity.

Proof: First we prove the hint, which is just integration by parts. Hence,

$$\begin{aligned}\int_0^1 v(x)^2 dx &= [xv(x)^2]_0^1 - 2\int_0^1 xv(x)v'(x)\,dx\\ &= v(1)^2 - 2\int_0^1 xv(x)v'(x)\,dx.\end{aligned}$$

So for the variant Poincarè inequality, consider, Couldn't figure out the proof with the hint. Used a similar idea though.

$$\begin{aligned}\|\sqrt{x}v\|^2_{L^2(0,1)} &= \int_0^1 xv^2\,dx\\ &= \frac{1}{2}x^2v^2\Big|_0^1 - \int_0^1 \frac{1}{2}x^2\cdot 2vv'\,dx\\ &= \frac{1}{2}v^2(1) - \int_0^1 x^2vv'\,dx\end{aligned}$$

Using the fact that $x^2 \le x$ on $[0,1]$, we have,

$$\begin{aligned}\|\sqrt{x}v\|^2_{L^2(0,1)} &\le \frac{1}{2}v^2(1) + \int_0^1 x|v||v'|\,dx\\ &\le \frac{1}{2}v^2(1) + \left(\int_0^1 xv^2\,dx\right)^{1/2}\left(\int_0^1 x(v')^2\,dx\right)^{1/2}\\ &= \frac{1}{2}v^2(1) + \|\sqrt{x}v\|_{L^2(0,1)}\|\sqrt{x}v'\|_{L^2(0,1)}\\ &\le \frac{1}{2}v^2(1) + \frac{1}{2}\|\sqrt{x}v\|^2_{L^2(0,1)} + \frac{1}{2}\|\sqrt{x}v'\|^2_{L^2(0,1)}\end{aligned}$$

Subtracting both sides by $\frac{1}{2}\|\sqrt{x}v\|^2_{L^2(0,1)}$, and then multiplying by 2, we have,

$$\|\sqrt{x}v\|^2_{L^2(0,1)} \le v^2(1) + \|\sqrt{x}v'\|^2_{L^2(0,1)}.$$

The result follows by taking the square root.

Now to prove that this variant Poincarè inequality holds for all $v \in V$. Let $v \in V$, then there exists a sequence $\{v_n\} \subset C^1([0,1])$ such that $v_n \to v$ in V since $C^1([0,1])$ is dense in V. That is, $\lim_{n\to\infty} \|v_n - v\|_V = 0$. Note that $\|\sqrt{x}v'\|_{L^2(0,1)} \leq \|v\|_V$ and $\|\sqrt{x}v\|_{L^2(0,1)} \leq \|v\|_V$, which implies that,

$$\lim_{n\to\infty} \|\sqrt{x}(v_n - v)\|_{L^2(0,1)} = \lim_{n\to\infty} \|\sqrt{x}(v_n - v)'\|_{L^2(0,1)} = 0. \tag{5}$$

Next, we claim that $v_n(1) \to v(1)$. To show this, we need a trace-type inequality. Specifically, we claim that $v(1) \leq C\|v\|_V$. Using the variation of the hint that we derived at the beginning, we have the following,

$$\begin{aligned} v^2(1) &= 2\int_0^1 xv^2\,dx + 2\int_0^1 x^2vv'\,dx \\ &\leq 2\Big(\int_0^1 (\sqrt{x}v)^2\,dx + \int_0^1 (\sqrt{x}v)(\sqrt{x}v')\,dx\Big) \\ &\leq 2(\|\sqrt{x}v\|^2_{L^2(0,1)} + \|\sqrt{x}v\|_{L^2(0,1)}\|\sqrt{x}v'\|_{L^2(0,1)}) \\ &\leq 4\|v\|^2_V \end{aligned}$$

Therefore,

$$|v_n(1) - v(1)| \leq |(v_n - v)(1)| \leq 2\|v_n - v\|_V \to 0. \tag{6}$$

Thus $v_n(1) \to v(1)$ as $n \to \infty$. All of this is to say that, the variant Poincarè inequality holds for all $v \in V$.

Now to show coercivity, lets start with $v \in C^1([0,1])$, then we have,

$$\begin{aligned} a(v,v) &= \int_0^1 xK(x)(v'(x))^2 + xq(x)v^2(x)\,dx + v^2(1) \\ &\geq \int_0^1 \kappa_0(\sqrt{x}v'(x))^2 + xq(x)v^2(x)\,dx + v^2(1) \\ &\geq \kappa_0\|\sqrt{x}v'\|^2_{L^2(0,1)} + v^2(1) \\ &\geq \min\{\kappa_0, 1\}(\|\sqrt{x}v'\|^2_{L^2(0,1)} + v^2(1)) \end{aligned}$$

We now apply the variant Poincarè inequality,

$$\begin{aligned} a(v,v) &\geq \min\{\kappa_0, 1\}\Big(\frac{1}{2}\|\sqrt{x}v'\|^2_{L^2(0,1)} + \frac{1}{2}v^2(1) + \frac{1}{2}\|\sqrt{x}v'\|^2_{L^2(0,1)} + \frac{1}{2}v^2(1)\Big) \\ &\geq \min\{\kappa_0, 1\}\Big(\frac{1}{2}\|\sqrt{x}v'\|^2_{L^2(0,1)} + \frac{1}{2}v^2(1) + \frac{1}{2}\|\sqrt{x}v\|^2_{L^2(0,1)}\Big) \\ &\geq \frac{1}{2}\min\{\kappa_0, 1\}\|v\|^2_V. \end{aligned}$$

■

c. Choose an integer $N \geq 2$, set $h = 1/N$, let $x_i = ih$, $0 \leq i \leq N$ and define the finite element space,

$$V_h = \{v_h \in C^0([0,1]);\ v_h|_{(x_i,x_{i+1})} \in \mathbb{P}_1, 0 \leq i \leq N-1\}. \tag{7}$$

Show that V_h is a subspace of V. Discretize the variational problem in this space. Prove existence and uniqueness of the discrete solution and establish an error estimate without estimating the norms of the interpolation errors.

Proof: Obviously the scalar multiplication and addition of any two vectors in V_h still belongs to V_h, since elements in V_h are piecewise linear. To see that $V_h \subset V$, note that for $v_h \in V_h$, v_h is piecewise linear so $\sqrt{x}v_h \in L^2(0,1)$ and $v_h \in L^2_{\text{loc}}$. Note we consider the derivative of v_h, written as v_h', to be the weak derivative. In particular this weak derivative will be a piecewise constant function with a finite number of discontinuities. Certainly then $v_h' \in L^2(0,1)$ and therefore $\sqrt{x}v_h'$ is also in $L^2(0,1)$.

Our variational problem then becomes, find $u_h \in V_h$ such that $a(u_h, v_h) = F(v_h)$ for all $v_h \in V_h$. Note also that V_h is a finite dimensional space, and therefore V_h is a closed subspace of V. We wish to apply Lax-Milgram for this finite dimensional problem, however we still need to show that $a(\cdot,\cdot)$ and $F(\cdot)$ are continuous. It is an easy check to show F is continuous, so we will only show that a is continuous. Consider,

$$
\begin{aligned}
|a(u,v)| &\le \int_0^1 K(x)|\sqrt{x}u'||\sqrt{x}v'| + q(x)|\sqrt{x}u||\sqrt{x}v|\,dx + |u(1)||v(1)| && (8)\\
&\le \max\{\|K\|_{L^\infty(0,1)}, \|q\|_{L^\infty(0,1)}\}\big(\|\sqrt{x}u'\|_{L^2(0,1)}\|\sqrt{x}v'\|_{L^2(0,1)} && (9)\\
&\qquad + \|\sqrt{x}u\|_{L^2(0,1)}\|\sqrt{x}v\|_{L^2(0,1)}\big) + 4\|u\|_V\|v\|_V && (10)\\
&\le \max\{4, \|K\|_{L^\infty(0,1)}, \|q\|_{L^\infty(0,1)}\}\|u\|_V\|v\|_V. && (11)
\end{aligned}
$$

Thus, Lax-Milgram applies to the variational problem on V_h which guarantees existence and uniqueness.

For the error estimates, we have by Cea's lemma that,

$$\|u - u_h\|_V \le \inf_{v_h \in V_h} \|u - v_h\|_V \le \|u - \mathcal{I}_h u\|_V,$$

where $\mathcal{I}_h$ is the canonical interpolation operator. (Note: you will need to actually prove Cea's lemma.) ■

Problem 2. Let Ω be a bounded domain in $\mathbb{R}^2$ with polygonal boundary $\partial\Omega$. Let

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v(x) = 0 \;\forall x \in \partial\Omega\} \tag{12}$$

be the standard Sobolev space of functions defined on Ω that vanish on the boundary.

In all that follows $T > 0$ is a given final time, $c > 0$ is a constant and $u_0 \in C^0(\Omega)$ are given functions. Consider the parabolic equation: Find a function u defined a.e. in $\Omega \times (0,T)$ solution of

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + cu = 0 \quad \text{a. e. in } \Omega \times (0,T)$$
$$u(x,t) = 0 \quad \text{a.e. in } \partial\Omega \times (0,T) \tag{13}$$
$$u(x,0) = u_0(x) \quad \text{a.e. in } \Omega.$$

Accept as a fact that problem (13) has one and only one solution u in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.

Let $\mathcal{T}_h$ be a finite element partition of Ω into triangles τ of diameter $h_\tau \le h$. Further, let

$$W_h = \{v_h \in C^0(\overline{\Omega}) \,:\, \forall \tau \in \mathcal{T}_h,\ v_h|_\tau \in \mathcal{P}_1,\ v_h|_{\partial\Omega} = 0\}, \tag{14}$$

be a finite element space of continuous piecewise linear functions over $\mathcal{T}_h$.

Consider the fully discrete backward Euler implicit approximation of (13): for K a positive integer, set $k = T/K$, define $t_n = nk$, $0 \le n \le K$, and for each $0 \le n \le K-1$, knowing $u_h^n \in W_h$ find $u_h^{n+1} \in W_h$ such that for all $v_h \in W_h$,

$$\frac{1}{k}(u_h^{n+1} - u_h^n, v_h) + a(u_h^{n+1}, v_h) = 0, \tag{15}$$

for $n = 0, 1, \ldots, K$ and $u_h^0 = I_h(u_0)$. Here $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, the bilinear form $a(u_h^{n+1}, v_h)$ comes from the variational formulation of problem (13), and I_h is the lagrange interpolation operator in W_h. Write the expression of $a(u_h^{n+1}, v_h)$.

a. Show that (15) defines a unique function u_h^{n+1} in W_h.

Proof: We start by first writing out the variational form. Using the backward Euler method, we express $\partial u_h/\partial t$ as a finite difference, that is,

$$\frac{\partial u_h}{\partial t} \approx \frac{u_h^{n+1} - u_h^n}{k}. \tag{16}$$

So, multiplying (13) by v_h and integrating over Ω, we have

$$\int_\Omega \Big(\frac{u_h^{n+1} - u_h^n}{k}\Big) v_h - \Delta u_h^{n+1} v_h + c u_h^{n+1} v_h \, dx = \int_\Omega \frac{1}{k}(u_h^{n+1} - u_h^n) v_h + \nabla u_h^{n+1} \cdot \nabla v_h + c u_h^{n+1} v_h \, dx.$$

So we have that $a(u_h^{n+1}, v_h) = \int_\Omega \nabla u_h^{n+1} \cdot \nabla v_h + c u_h^{n+1} v_h \, dx$.

The proof for existence and uniqueness is inductive; that is, we assume that the solution has been computed up to time, t^n. This discrete variational problem can be formulated as a matrix equation by expressing our functions in terms of the nodal Lagrange basis functions. In particular, we write $u_h^{n+1} = \sum_{i=1}^N u_i^{n+1} \phi_i$ where $u_i^{n+1} \in \mathbb{R}$ are the unknown coefficients and ϕ_i are the nodal Lagrange basis functions. We also write $u_h^n = \sum_{i=1}^N u_i^n \phi_i(x)$ and set $v_h = \phi_j$ in the discrete variational equation. Doing so, gives us an $N \times N$ linear system of equations:

$$(\mathbf{M} + k\mathbf{A})\mathbf{U}^{n+1} = \mathbf{U}^n, \tag{17}$$

where $\mathbf{U}^n = (u_1^n, \dots, u_N^n)^T$ and M and A are matrices with entries (ϕ_i, ϕ_j) and $a(\phi_i, \phi_j)$, respectively.

First consider the problem in which $\mathbf{U}^n = \mathbf{0}$. We claim that $\mathbf{U}^{n+1} = \mathbf{0}$ is the only solution. This matrix equation is now equivalent to the variational problem of solving $\frac{1}{k}(u_h^{n+1}, v_h) + a(u_h^{n+1}, v_h) = 0$. Testing this equation with $v_h = u_h^{n+1}$, we arrive at the following conclusion,

$$\frac{1}{k}\|u_h^{n+1}\|_{L^2(\Omega)}^2 + |u_h^{n+1}|_{H^1(\Omega)}^2 + c\|u_h^{n+1}\|_{L^2(\Omega)}^2 = 0. \tag{18}$$

This implies that $u_h^{n+1} \equiv 0$, i.e. $\mathbf{U}^{n+1} = \mathbf{0}$, which means that the null space of our operator $M + kA$ is trivial. By the rank-nullity theorem, this implies that the operator $M + kA$ has full rank. Which ultimately concludes there exists a unique solution to our finite dimensional variational problem. ■

b. Prove the following stability estimate,

$$\sup_{1 \le n \le K} \|u_h^n\|_{L^2(\Omega)}^2 + k\sum_{n=1}^{K} |u_h^n|_{H^1(\Omega)}^2 \le \|u_h^0\|_{L^2(\Omega)}^2 \tag{19}$$

Proof: Let n be arbitrary in $\mathbb{N}$. Then let $v_h = u_h^{n+1}$ in our backward Euler method. This gives us,

$$\frac{1}{k}(u_h^{n+1} - u_h^n, u_h^{n+1}) + a(u_h^{n+1}, u_h^{n+1}) = 0,$$

which can be written as,

$$\begin{aligned}(u_h^{n+1}, u_h^{n+1}) &= (u_h^n, u_h^{n+1}) - ka(u_h^{n+1}, u_h^{n+1}) \\ &\le \|u_h^n\|_{L^2(\Omega)}\|u_h^{n+1}\|_{L^2(\Omega)} - k|u_h^{n+1}|_{H^1(\Omega)}^2 - ck\|u_h^{n+1}\|_{L^2(\Omega)}^2 \\ &\le \frac{1}{2}\|u_h^n\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u_h^{n+1}\|_{L^2(\Omega)}^2 - k|u_h^{n+1}|_{H^1(\Omega)}^2.\end{aligned}$$

Subtracting both sides by $\frac{1}{2}\|u_h^{n+1}\|_{L^2(\Omega)}^2$, we have the inequality,

$$\frac{1}{2}\|u_h^{n+1}\|_{L^2(\Omega)}^2 \le \frac{1}{2}\|u_h^n\|_{L^2(\Omega)}^2 - k|u_h^{n+1}|_{H^1(\Omega)}^2.$$

We can then repeatedly apply the inequality for u_h^n, u_h^{n-1}, etc... Hence, we have,

$$\frac{1}{2}\|u_h^{n+1}\|_{L^2(\Omega)}^2 \le \frac{1}{2}\|u_h^0\|_{L^2(\Omega)}^2 - k\sum_{j=1}^{n+1}|u_h^j|_{H^1(\Omega)}^2 \le \frac{1}{2}\|u_h^0\|_{L^2(\Omega)}^2 - \frac{1}{2}k\sum_{j=1}^{n+1}|u_h^j|_{H^1(\Omega)}^2.$$

Rearranging the equation we have,

$$\|u_h^{n+1}\|_{L^2(\Omega)}^2 + k\sum_{j=1}^{n+1}|u_h^j|_{H^1(\Omega)}^2 \le \|u_h^0\|_{L^2(\Omega)}^2.$$

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c. Also prove the estimate

$$\sup_{1\le n\le K}|u_h^n|_{H^1(\Omega)} \le |u_h^0|_{H^1(\Omega)}. \tag{20}$$

Proof: We want to make a "guess" for the test function v_h in the discrete equation to arrive at our estimate. Let us define the discrete Laplacian operator $A_h : W_h \to W_h$ to be the action

$$(A_h v_h, w_h) = \int_\Omega \nabla v_h \cdot \nabla w_h \, dx. \tag{21}$$

The existance of this operator can be proven through linear algebra. Let us recall our discrete equation:

$$\frac{1}{k}(u_h^{n+1} - u_h^n, v_h) + a(u_h^{n+1}, v_h) = 0. \tag{22}$$

Expanding $a(\cdot,\cdot)$ using the inner product notation yields:

$$\frac{1}{k}(u_h^{n+1} - u_h^n, v_h) + (\nabla u_h^{n+1}, \nabla v_h) + c(u_h^{n+1}, v_h) = 0 \tag{23}$$

Note that the second term can written as follows: $(\nabla u_h^{n+1}, \nabla v_h) = (A_h u_h^{n+1}, v_h)$. Now let us choose $v_h = A_h u_h^{n+1}$ and substitute into the above equation:

$$\frac{1}{k}(u_h^{n+1} - u_h^n, A_h u_h^{n+1}) + (A_h u_h^{n+1}, A_h u_h^{n+1}) + c(u_h^{n+1}, A_h u_h^{n+1}) = 0. \tag{24}$$

But note that $c(u_h^{n+1}, A_h u_h^{n+1}) = c\int_\Omega |\nabla u_h^{n+1}|^2 \, dx$ and $(A_h u_h^{n+1}, A_h u_h^{n+1})$ are both nonnegative. So we can drop those two terms to arrive at the inequality, $\frac{1}{k}(u_h^{n+1} - u_h^n, A_h u_h^{n+1}) \le 0$. Therefore,

$$\begin{aligned}
|u_h^{n+1}|_{H^1(\Omega)}^2 &= \int_\Omega |\nabla u_h^{n+1}|^2 \, dx \\
&= (A_h u_h^{n+1}, u_h^{n+1}) \\
&\le (A_h u_h^{n+1}, u_h^n) \\
&= \int_\Omega \nabla u_h^{n+1} \cdot \nabla u_h^n \, dx \\
&\le |u_h^{n+1}|_{H^1(\Omega)} |u_h^n|_{H^1(\Omega)}.
\end{aligned}$$

Thus, $|u_h^{n+1}|_{H^1(\Omega)} \le |u_h^n|_{H^1(\Omega)}$. Applying this inequality for each time step n and then taking the supremum yields the result. ∎

Problem 3. Consider the interval $(0,1)$ and the set of continuous functions $\hat{v}$ defined on $[0,1]$. Let $\hat{a}_1 = 0$, $\hat{a}_2 = \frac{1}{2}$, $\hat{a}_3 = 1$.

a. Consider the following two sets of degrees of freedom,

$$\Sigma_1 = \{\hat{v}(\hat{a}_j),\ j = 1,2,3\} \quad \text{and} \quad \Sigma_2 = \{\hat{v}(\hat{a}_1), \hat{v}(\hat{a}_3), \int_0^1 \hat{v}(s)\, ds\}. \tag{25}$$

Write down the basis functions of $\mathcal{P}_2$ (for both sets of degrees of freedom) such that

1. $p_i \in \mathcal{P}_2$, $1 \le i \le 3$, satisfying: $p_i(\hat{a}_j) = \delta_{i,j}$, $1 \le i,j \le 3$ for the set Σ_1;
2. $q_i \in \mathcal{P}_2$, $1 \le i \le 3$, satisfying:

$$q_i(\hat{a}_j) = \delta_{i,j} \quad \text{and} \quad \int_0^1 q_i(s)\, ds = 0, \tag{26}$$

for $i = 1,3$, and $j = 1,3$ and

$$\int_0^1 q_2(s)\, ds = 1, \quad \text{and } q_2(\hat{a}_1) = q_2(\hat{a}_3) = 0. \tag{27}$$

In both cases, write down the FE interpolant $\hat{\Pi}(\hat{w})$ of a given function $\hat{w} \in C^0([0,1])$.

Proof: Lets start with Σ_1 first. So if $p_1(\hat{a}_1) = 1$ and $p_1(\hat{a}_2) = p_1(\hat{a}_3) = 0$, then this implies that $p_1(x) = c(x - 1/2)(x-1)$. Plugging in $\hat{a}_1$ we can find c; $p_1(0) = c(-1/2)(-1) = 1$, hence $c = 2$. We can repeat this process for finding p_2 and p_3, which gives us,

$$\begin{aligned} p_1(x) &= 2(x - \frac{1}{2})(x-1) \\ p_2(x) &= -4x(x-1) \\ p_3(x) &= 2x(x - \frac{1}{2}). \end{aligned}$$

Now for Σ_2, we have $q_1(0) = 1$, $\int_0^1 q_1(s)\, ds = 0$ and $q_1(1) = 0$. So if $q_1(x) = ax^2 + bx + c$, then $q_1(0) = 1$ implies $c = 1$. For the other two conditions, we have, $q_1(1) = a + b + 1 = 0$ and $\int_0^1 q_1(s)\, ds = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + x]_0^1 = \frac{1}{3}a + \frac{1}{2}b + 1 = 0$. Solving this system of equations, we find $b = -4$ and $a = 3$. Again, we repeat this method for q_2 and q_3 to find,

$$\begin{aligned} q_1(x) &= 3x^2 - 4x + 1 \\ q_2(x) &= -6x(x-1) \\ q_3(x) &= 3x^2 - 2x. \end{aligned}$$

For the FE interpolant using the degrees of freedom, Σ_1, we have,

$$\begin{aligned}\hat{\Pi}_1(\hat{w}) &= \hat{w}(\hat{a}_1)p_1(x) + \hat{w}(\hat{a}_2)p_2(x) + \hat{w}(\hat{a}_3)p_3(x)\\ &= 2\hat{w}(0)(x-1/2)(x-1) - 4\hat{w}(1/2)x(x-1) + 2\hat{w}(1)x(x-1/2).\end{aligned}$$

Similarly, for Σ_2, we have,

$$\hat{\Pi}_2(\hat{w}) = \hat{w}(0)(3x^2-4x+1) - 6\Big(\int_0^1 \hat{w}(s)\,ds\Big)x(x-1) + 2\hat{w}(1)x(x-1/2).$$

b. Consider the interval $[a,b]$, let F map $[0,1]$ onto $[a,b]$, and let v be given in $H^3(a,b)$. Define $\Pi(v)$ by $(\Pi(v))\circ F = \hat{\Pi}(v\circ F)$. Give the Bramble-Hilbert argument to get an estimate in terms of $h = b-a$ for the error

$$\|v' - \Pi(v)'\|_{L^2(a,b)}. \tag{28}$$

Explain how to modify the proof when v is less regular, e.g. $v \in H^2(a,b)$.

Proof: We first transfer the integral over $[a,b]$ to $[0,1]$ and apply Bramble-Hilbert lemma. We can define F as $F(\xi) = h\xi + a$.

Let F map $[0,1]$ onto $[a,b]$ be explicitly defined by $F(\xi) = h\xi + a$ with $\det(F') = h$ where $h = b-a$. Let $v \in H^3(a,b)$. Define $\Pi(v)$ by $(\Pi(v))\circ F = \widehat{\Pi}(v\circ F)$. Let $\widehat{\Pi}$ be either $\widehat{\Pi}_1$ or $\widehat{\Pi}_2$ from part (a). This relationship of Π and $\widehat{\Pi}$ can be seen with the following diagram

$$\begin{array}{ccc} H^3(a,b) & \xrightarrow{\ \Pi\ } & \mathbb{P}_2 \\ \psi\big\downarrow & & \big\downarrow\psi \\ H^3(0,1) & \xrightarrow{\ \widehat{\Pi}\ } & P \end{array}$$

We want to compute the norm $\|v' - \Pi(v)'\|_{L^2(a,b)}$ on the interval $[0,1]$. First note that by the chain rule

$$\frac{d}{dx} = \frac{d\xi}{dx}\frac{d}{d\xi} = \frac{1}{h}\frac{d}{d\xi}.$$

Then consider,

$$\begin{aligned}\|v' - \Pi(v)'\|^2_{L^2(a,b)} &= \int_a^b \Big|\frac{d}{dx}(v(x) - \Pi(v)(x))\Big|^2\,dx\\ &= \int_0^1 \Big|\frac{1}{h}\frac{d}{d\xi}\big((v\circ F)(\xi) - \hat{\Pi}(v\circ F)(\xi)\big)\Big|^2 h\,d\xi.\end{aligned}$$

If we let $\hat{v} = v \circ F$, then we can write,

$$\begin{aligned}
\|v' - \Pi(v)'\|^2_{L^2(a,b)} &= \frac{1}{h}\int_0^1 \left|\frac{d}{d\xi}\big(\hat{v} - \hat{\Pi}(\hat{v})\big)\right|^2 d\xi \\
&= \frac{1}{h}\int_0^1 \left|\frac{d}{d\xi}\big((\mathrm{Id} - \hat{\Pi})(\hat{v})\big)\right|^2 d\xi \\
&= \frac{1}{h}|(\mathrm{Id} - \hat{\Pi})(\hat{v})|^2_{H^1(0,1)}.
\end{aligned}$$

Then notice that $|(\mathrm{Id} - \hat{\Pi})(\cdot)|_{H^1(0,1)}$ is a sublinear functional which is exactly zero for all $\hat{v} \in \mathcal{P}_2$. So by the Bramble-Hilbert lemma, there exists a constant c such that $|(\mathrm{Id} - \hat{\Pi})(\hat{v})|_{H^1(0,1)} \leq c|\hat{v}|_{H^3(0,1)}$. Therefore, by the Bramble-Hilbert lemma,

$$\begin{aligned}
\|v' - \Pi(v)'\|^2_{L^2(a,b)} &\leq \frac{1}{h}c\int_0^1 \left|\frac{d^3}{d\xi^3}\hat{v}\right|^2 d\xi \\
&= \frac{c}{h}\int_a^b \left|h^3\frac{d^3}{dx^3}v\right|^2 \frac{1}{h}\,dx \\
&= ch^4\int_a^b \left|\frac{d^3}{dx^3}v\right|^2 dx \\
&= ch^4|v|^2_{H^3(a,b)}.
\end{aligned}$$

Thus we have,

$$\|v' - \Pi(v)'\|_{L^2(a,b)} \leq ch^2|v|_{H^3(a,b)}.$$

Let us now consider the case where we have lower regularity on v, that is, we assume $v \in H^2(a,b)$. Let us assume that $\widehat{\Pi} = \widehat{\Pi}_1$. The goal is to redo the previous proof and modify it appropriately for when we have lower regularity. We now repeat the above arguments for functions in $H^2(\Omega)$. So we have that

$$\begin{aligned}
\|v' - \Pi(v)'\|^2_{L^2(a,b)} &= \frac{1}{h}\int_0^1 \left|\frac{d}{d\xi}\big(\hat{v} - \widehat{\Pi}(\hat{v})\big)\right|^2 d\xi \\
&= \frac{1}{h}\int_0^1 \left|\frac{d}{d\xi}\big((\mathrm{Id} - \widehat{\Pi})(\hat{v})\big)\right|^2 d\xi \\
&= \frac{1}{h}|(\mathrm{Id} - \widehat{\Pi})(\hat{v})|^2_{H^1(0,1)}
\end{aligned}$$

Then, applying the Bramble-Hilbert lemma yields

$$\begin{aligned}
\|v' - \Pi(v)'\|^2_{L^2(a,b)} &\leq \frac{1}{h}c\int_0^1 \left|\frac{d^2}{d\xi^2}\hat{v}\right|^2 d\xi \\
&= \frac{c}{h}\int_a^b \left|h^2\frac{d^2}{dx^2}v\right|^2 \frac{1}{h}\,dx \\
&= ch^2\int_a^b \left|\frac{d^2}{dx^2}v\right|^2 dx \\
&= ch^2|v|^2_{H^1(a,b)}
\end{aligned}$$

Thus our new estimate is given,

$$\|v' - \Pi(v)'\|_{L^2(a,b)} \leq ch|v|_{H^2(a,b)}. \tag{29}$$

■