Applied/Numerical Qualifier Solution: January 2012

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Problem 1. Let $\Omega = (0,1) \times (0,1)$, $f \in C^0(\Omega)$ and $q \in \mathbb{R}$ with $q \geq 0$. Consider the boundary value problem

$$-\Delta u + qu = f \text{ in } \Omega; \tag{1}$$

$$u = 0 \text{ on } \partial\Omega. \tag{2}$$

We are interested in approximating the quantity $\alpha := \int_{\partial\Omega} \mathbf{n} \cdot \nabla u$ where $\mathbf{n}$ is the outward unit normal of Ω.

a. The boundary problem has a weak formulation: Find $u \in V$ such that $\forall v \in V$

$$a(u,v) = L(v). \tag{3}$$

Identify V, $a(u,v)$ and $L(v)$. Show that there exists a unique solution $u \in V$ satisfying the above weak formulation.

Solution: We multiply equation (1) by some $v \in V$ and integrate over Ω. So we have,

$$\int_\Omega -\Delta uv + quv \, dx = \int_\Omega \nabla u \cdot \nabla v + quv \, dx - \int_{\partial\Omega} (\mathbf{n} \cdot \nabla u) v \, ds.$$

Since we don't know anything about the normal derivative on the boundary, we require $v|_{\partial\Omega} = 0$. Then our space is $V = H_0^1(\Omega)$ with the usual H^1-norm and

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + quv \, dx$$

$$L(v) = \int_\Omega fv \, dx$$

It is easy to check that a and L are both continuous. To prove coercivity, we invoke a Poincarè inequality: $||u||_{L^2(\Omega)} \leq C|u|_{H^1(\Omega)}$, without proof as we have that $u|_{\partial\Omega} \equiv 0$. Coercivity follows,

$$\begin{aligned} a(u,u) &= \int_\Omega |\nabla u|^2 + qu^2 \, dx \\ &\geq \tfrac{1}{2}|u|^2_{H^1(\Omega)} + \tfrac{1}{2}|u|^2_{H^1(\Omega)} \\ &\leq \tfrac{1}{2}\min\{1, \tfrac{1}{C}\}||u||^2_{H^1(\Omega)} \end{aligned}$$

So by the Lax-Milgram lemma, there exists a unique solution $u \in V$ to our variational equation. ■

b. Let $\{\mathcal{T}_h\}_{0<h<1}$ be a sequence of conforming shape-regular subdivisions of Ω such that $\text{diam}(T) \leq h$, for all $T \in \mathcal{T}_h$ and define

$$V_h := \{v \in C^0(\Omega) \cap V \mid \forall T \in \mathcal{T}_h,\ v|_T \text{ is linear }\} \tag{4}$$

Write the weak formulation satisfied by the finite element approximation $u_h \in V_h$ of u. Prove that the function u_h exists and is unique.

Solution: The weak formulation is, find $u_h \in V_h$ such that $a(u_h, v_h) = L(v_h)$ for all $v_h \in V_h \cap H^1_0(\Omega)$. Since V_h is a subspace of V, Lax-Milgram applies again. ■

c. Assume from now that $u \in H^2(\Omega)$. Derive the error estimate

$$||u - u_h||_{H^1(\Omega)} \leq c_1 h ||u||_{H^2(\Omega)}, \tag{5}$$

where c_1 is a constant independent of h and u. Hint: you can use without proof the fact that there exists a constant C independent of h such that for any $v \in H^2(\Omega)$

$$\inf_{v_h \in V_h} ||v - v_h||_V \leq Ch||v||_{H^2(\Omega)}. \tag{6}$$

Solutions: By Cea's lemma, we have

$$||u - u_h||_{H^1(\Omega)} \leq \inf_{v_h \in V_h} ||u - v_h||_{H^1(\Omega)} \leq Ch||u||_{H^2(\Omega)}. \tag{7}$$

You will need to actually prove Cea's lemma. ■

d. Show that for the constant function $w(x) = 1$ we have

$$\alpha = a(u, w) - L(w). \tag{8}$$

Now let $\alpha_h := a(u_h, w) - L(w)$. Using the previous parts, show that when $q > 0$ there holds

$$|\alpha - \alpha_h| \le c_2 h^2 \|u\|_{H^2(\Omega)}, \tag{9}$$

where c_2 is a constant independent of h and u. What can you say about $|\alpha - \alpha_h|$ when $q = 0$?

Solution: Note that $w \notin H_0^1(\Omega)$. Additionally, since $u \in H^2(\Omega)$, u will also be a strong solution, i.e., $-\Delta u + qu = f$ in Ω. So to start, consider,

$$\begin{aligned} a(u,1) - L(1) &= \int_\Omega \nabla u \cdot \nabla 1 + qu \, dx - \int_\Omega f \, dx \\ &= \int_\Omega qu - f \, dx \\ &= \int_\Omega \Delta u \, dx \\ &= \int_{\partial\Omega} \mathbf{n} \cdot \nabla u \, ds. \end{aligned}$$

Note that a straightforward method will not get the correct degree of h, consider,

$$\begin{aligned} |\alpha - \alpha_h| &= |a(u,w) - L(w) - a(u_h, w) + L(w)| \\ &= |a(u - u_h, w)| \\ &\le \|u - u_h\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)} \\ &\le Ch\|u\|_{H^2(\Omega)}. \end{aligned}$$

So we need to look closer at the expressions of α and α_h. So consider,

$$\begin{aligned} |\alpha - \alpha_h| &= \left| \int_{\partial\Omega} \mathbf{n} \cdot \nabla u \, ds - \int_\Omega \nabla u_h \cdot \nabla 1 + qu_h \, dx + \int_\Omega f \, dx \right| \\ &= \left| \int_\Omega \Delta u \, dx - \int_\Omega qu_h \, dx + \int_\Omega f \, dx \right| \\ &= \left| \int_\Omega qu \, dx - \int_\Omega qu_h \, dx \right| \\ &\le q \int_\Omega |u - u_h| \, dx \\ &\le q\|u - u_h\|_{L^2(\Omega)}. \end{aligned}$$

Now we need to prove an L^2 error estimate which we will do so using the Aubin-Nitsche trick. Consider the dual problem, find $z \in H_0^1(\Omega)$ such that

$$a(v, z) = (u - u_h, v),$$

for all $v \in H_0^1(\Omega)$. We assume full regularity, that is, $\|u\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$. So in our dual problem, we have $\|z\|_{H^2(\Omega)} \le C\|u - u_h\|_{L^2(\Omega)}$. Now, in our dual problem, take $v = u - u_h$ and apply Galerkin orthogonality to write,

$$\|u - u_h\|_{L^2(\Omega)}^2 = (u - u_h, u - u_h) = a(u - u_h, z) = a(u - u_h, z - v_h)$$

for all $v_h \in V_h$. So in particular, we can take $v_h := \Pi_h z$ (the projection of z onto the space V_h). Then applying the continuity of a the result from part c. and the regularity assumption, we have,

$$\begin{aligned}\|u - u_h\|_{L^2(\Omega)}^2 &\le \|u - u_h\|_{H^1(\Omega)}\|z - \Pi_h z\|_{H^1(\Omega)} \\ &\le Ch\|u\|_{H^2(\Omega)} \cdot h\|z\|_{H^2(\Omega)} \\ &\le Ch^2\|u\|_{H^2(\Omega)}\|u - u_h\|_{L^2(\Omega)},\end{aligned}$$

where we have used the inequality, $\|z - \Pi_h z\|_{H^1(\Omega)} \le Ch\|z\|_{H^2(\Omega)}$. You might want to prove this projection inequality just to be on the safe side. But I don't want to repeat this proof which I've done in the other exams. Dividing both sides by $\|u - u_h\|_{L^2(\Omega)}$, and combining all of our results, we have,

$$\|u - u_h\|_{L^2(\Omega)} \le Ch^2\|u\|_{H^2(\Omega)}.$$

Hence

$$|\alpha - \alpha_h| \le Ch^2\|u\|_{H^2(\Omega)}.$$

Now if $q = 0$, then our bilinear form becomes, $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$. Computing $|\alpha - \alpha_h|$, we have,

$$|\alpha - \alpha_h| = |a(u, 1) - L(1) - a(u_h, 1) + L(1)| = 0,$$

since $\nabla 1 = 0$. Thus $\alpha = \alpha_h$ for $q = 0$. ■

Problem 2. Let K be a polyhedron in $\mathbb{R}^d$, $d \ge 1$. Let $h = \text{diam}(K)$ and define

$$\hat{K} = \{\hat{\mathbf{x}} = \mathbf{x}/\text{diam}(K), \mathbf{x} \in K\}. \tag{10}$$

Show that there exists a constant c solely depending on $\hat{K}$ such that for any $v \in H^1(K)$,

$$\|v\|_{L^2(\partial K)} \le c\left(h^{-1/2}\|v\|_{L^2(K)} + h^{1/2}\|\nabla v\|_{L^2(K)}\right). \tag{11}$$

Proof: We start by defining the map $T_K : \widehat{K} \to K$ defined by $\boldsymbol{x} = T_K(\hat{\boldsymbol{x}}) = h\hat{\boldsymbol{x}}$. Let $T_{e_K} : \hat{e} \to e_K$, with $\hat{e}$ and e_K being $d-1$ dimensional face of the elements $\widehat{K}$ and K, respectively. Specifically, T_{e_K} is defined by $T_K|_{\hat{e}}$. Let $\widehat{\mathcal{F}}$ and $\mathcal{F}_K$ denotes the faces of $\widehat{K}$ and K, respectively.

Then we separate the boundary of K into its $d-1$ dimensional sides, and perform a change of variable,

$$\begin{aligned} \|v\|^2_{L^2(\partial K)} &= \int_{\partial K} |v(s)|^2\, ds \\ &= \sum_{e\in\mathcal{F}_K} \int_e |v(s)|^2\, ds \\ &= \sum_{\hat{e}\in\widehat{\mathcal{F}}} \int_{\hat{e}} |(v\circ T_{e_K})(\hat{s})|^2 \frac{|e|}{|\hat{e}|}\, d\hat{s}. \end{aligned}$$

Note that $|K| \le Ch^d$ and $|e| \le Ch^{d-1}$. Hence $|\partial K| = \sum_{e\in\mathcal{F}_K} |e| \le Ch^{d-1}$. Now let $\hat{v} := v\circ T_K$, and since $v \in H^1(K)$, we can apply the trace inequality, $(\|v\|_{L^2(\partial\widehat{K})} \le C\|v\|_{H^1(\widehat{K})})$, therefore,

$$\begin{aligned} \|v\|^2_{L^2(\partial K)} &\le Ch^{d-1} \sum_{\hat{e}\in\widehat{\mathcal{F}}} \|\hat{v}\|^2_{L^2(\hat{e})} \\ &= Ch^{d-1}\|\hat{v}\|^2_{L^2(\partial\widehat{K})} \\ &\le Ch^{d-1}\|\hat{v}\|^2_{H^1(\widehat{K})} \\ &= Ch^{d-1}\Big(\int_{\widehat{K}} |\hat{v}|^2\, d\hat{\boldsymbol{x}} + \int_{\widehat{K}} |\widehat{\nabla}\hat{v}|^2\, d\hat{\boldsymbol{x}}\Big) \\ &\le Ch^{d-1}\frac{|\widehat{K}|}{|K|}\Big(\int_K |v|^2\, d\boldsymbol{x} + \int_K h^2|\nabla v|^2\, d\boldsymbol{x}\Big) \\ &\le Ch^{-1}\Big(\int_K |v|^2\, d\boldsymbol{x} + \int_K h^2|\nabla v|^2\, d\boldsymbol{x}\Big) \\ &= C(h^{-1}\|v\|^2_{L^2(K)} + h|v|^2_{H^1(K)}). \end{aligned}$$

Note the change of variables for the gradient uses the fact that $\frac{\partial}{\partial\hat{x}_i} = \frac{\partial x_i}{\partial\hat{x}_i}\frac{\partial}{\partial x_i} = ch\frac{\partial}{\partial x_i}$ for $i = 1,\dots,d$ and some constant c. Taking the square root of both sides, and using the inequality, $\sqrt{a+b} \le \sqrt{a}+\sqrt{b}$, we have the final result. Note that we have simply used C to represent every constant rather than keeping track of each unique constant. ■

Problem 3: Let $u_0 : (0,1) \to \mathbb{R}$ be a given smooth initial condition and $T > 0$ be a given final time. Let $u : [0,T]\times\Omega \to \mathbb{R}$ be a smooth function satisfying $u(t,0) = u(t,1) = 0$ for any $t \in [0,T]$ and $\forall v \in C_c^\infty([0,T)\times(0,1))$:

$$\begin{aligned} -\int_0^T\int_0^1 u(t,x)v_t(t,x)\, dxdt - \int_0^1 u_0(x)v(0,x)\, dx \\ + \int_0^T\int_0^1 u_x(t,x)v_x(t,x)\, dxdt + \int_0^T\int_0^1 u(t,x)v(t,x)\, dxdt = 0 \end{aligned} \tag{12}$$

Here $C_c^\infty([0,T)\times(0,1))$ is the space of functions belonging to $C^\infty([0,T]\times[0,1])$ and compactly supported in $[0,T)\times(0,1)$.

a. Derive the corresponding strong formulation.

Solution: We integrate the two integrals in (12) which contain derivatives in them, by parts. This gives us

$$\int_0^T \int_0^1 (u_t(t,x) - u_{xx}(t,x) + u(t,x))v(t,x)\,dx\,dt = 0.$$

By the fundamental theorem of variational calculus, our strong formulation becomes,

$$\begin{cases} u_t - u_{xx} + u = 0, \quad \text{for } (t,x) \in (0,T] \times (0,1) \\ u(0,x) = u_0(x) \\ u(t,0) = u(t,1) = 0. \end{cases} \tag{13}$$

■

b. Let $N > 0$ be an integer, $h = 1/N$ and $x_n = nh$, $n = 0, ..., N$. Derive the semi-discrete approximation of (12) using continuous piecewise linear finite elements.

Solution: Define our discrete space by,

$$V_h := \{v_h \in C^0(0,1) : v_h|_{[x_{i-1},x_i]} \in \mathbb{P}_1,\ i = 1,2,\dots,N,\ v_h(0) = v_h(1) = 0\},$$

and equip it with the H^1-norm. Note that $V_h = \text{span}\{\varphi_i\}_{i=1}^N$, where φ_i are the usual Lagrange shape functions. Define,

$$a(u,v) := \int_0^T \int_0^1 u_x(t,x)v_x(t,x) + u(t,x)v(t,x)\,dx\,dt$$

and if we integrate the first integral in (12), we can formulate problem in a nicer way. Thus (12) becomes, $((u_h)_t, v_h) + a(u_h, v_h) = 0$.

Now for $t > 0$, we can write our semi-discrete solution as

$$u_h(t) = \sum_{i=1}^N U_i(t)\varphi_i(x), \tag{14}$$

where the $U_i(t)$ are the unknown time dependent coefficients. Thus our semi-discrete problem becomes: for $t > 0$, find $u_h(t) \in V_h$ such that

$$((u_h)_t, v_h) + a(u_h, v_h) = 0 \tag{15}$$

for all $v_h \in V_h$.

find $u_h \in L^2(0,T;V_h)$ such that

$$((u_h)_t, v_h) + a(u_h, v_h) = 0 \tag{16}$$

for all $v_h \in L^2(0,T;V_h)$.

Note quite sure how to formulate the problem statement.

c. In addition, let $M > 0$ be an integer, $\tau = T/M$ and $t_m = m\tau$ for $m = 0, \ldots, M$. Write the fully discrete schemes corresponding to backward Euler and forward Euler methods, respectively.

Solution: We define $u_h^m := u_h(t_m, x)$ where $u_h(t,x)$ is the semi-discrete representation of as in part b. So for the backward Euler, we have

$$\begin{cases} \text{Find } u_h^{m+1} \in V_h \text{ such that} \\ (\frac{u_h^{m+1} - u_h^m}{\tau}, v_h) + a(u_h^{m+1}, v_h) = 0 \quad \forall v_h \in V_h \\ \text{where } u_h^0 = u_{h,0} \end{cases}$$

The forward Euler, is defined similarly,

$$\begin{cases} \text{Find } u^{m+1} \in V_h \text{ such that} \\ (\frac{u_h^{m+1} - u_h^m}{\tau}, v_h) + a(u_h^m, v_h) = 0 \quad \forall v_h \in V_h \\ \text{where } u_h^0 = u_{h,0} \end{cases}$$

■

d. Prove that the backward (implicit) Euler scheme is unconditionally stable while the forward (explicit) Euler method is stable provided $\tau \le ch^2$, where c is a constant independent of h and τ.

Solution: To show unconditional stability of the backward Euler scheme, we test the scheme with $v_h = u_h^{m+1} - u_h^m$. So consider,

$$\begin{aligned} (\frac{u_h^{m+1} - u_h^m}{\tau}, u_h^{m+1} - u_h^m) + a(u_h^{m+1}, u_h^{m+1} - u_h^m) = 0 \\ \|u_h^{m+1} - u_h^m\|_{L^2}^2 + \tau a(u_h^{m+1}, u_h^{m+1} - u_h^m) = 0 \end{aligned}$$

Then, since $||u_h^{m+1} - u_h^m||_{L^2}^2 \geq 0$, we can drop that term and we are thus left with

$$\tau a(u_h^{m+1}, u_h^{m+1} - u_h^m) \leq 0 \quad \Leftrightarrow \quad a(u_h^{m+1}, u_h^{m+1}) \leq a(u_h^{m+1}, u_h^m)$$

Using continuity of $a(\cdot,\cdot)$ and the definition of $a(\cdot,\cdot)$ we have the following

$$||u_h^{m+1}||_{H^1}^2 \leq ||u_h^{m+1}||_{H^1}||u_h^m||_{H^1}$$

Therefore we can conclude that $||u_h^{m+1}||_{H^1} \leq ||u_h^m||_{H^1} \leq \cdots \leq ||u_{0,h}||_{H^1}$. I.e. the backward Euler is unconditionally stable.

For the forward Euler, we will test with $v_h = u_h^{m+1}$, so we have,

$$\Big(\frac{u_h^{m+1} - u_n^m}{\tau}, u_h^{m+1}\Big) + a(u_h^m, u_h^{m+1}) = 0.$$

Rearranging this, we then apply Cauchy-Schwarz and the inverse inequality to get,

$$\begin{aligned}
||u_h^{m+1}||_{L^2(0,1)}^2 &= (u_h^m, u_h^{m+1}) - \tau a(u_h^m, u_h^{m+1}) \\
&= (1-\tau)(u_h^m, u_h^{m+1}) - \tau((u_h^m)_x, (u_h^{m+1})_x) \\
&\leq |1-\tau|||u_h^m||_{L^2(0,1)}||u_h^{m+1}||_{L^2(0,1)} + \tau|u_h^m|_{H^1(0,1)}|u_h^{m+1}|_{H^1(0,1)} \\
&\leq |1-\tau|||u_h^m||_{L^2(0,1)}||u_h^{m+1}||_{L^2(0,1)} + \frac{C}{h^2}\tau||u_h^m||_{L^2(0,1)}||u_h^{m+1}||_{L^2(0,1)}.
\end{aligned}$$

Thus we have,

$$||u_h^{m+1}||_{L^2(0,1)} \leq (|1-\tau| + \frac{C}{h^2}\tau)||u_h^m||_{L^2(0,1)}.$$

Therefore, we will have stability if $|1-\tau| + C\tau/h^2 \leq 1$, hence if $0 \leq |1-\tau| \leq 1 - C\tau/h^2$. Which confirms the CFL condition that,

$$\tau \leq Ch^2.$$

■