Applied/Numerical Qualifier Solution: January 2013

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Problem 1.

a. You may assume the inequality

$$\|u\|^2_{H^1(\hat{\tau})} \leq C\left(\int_{\hat{\tau}} |\nabla u|^2 \, d\hat{x} + \bar{u}^2\right), \quad \text{for all } u \in H^1(\hat{\tau}).$$

Here $\hat{\tau}$ is the reference triangle in $\mathbb{R}^2$, $\bar{u}$ denotes the mean value of u on $\hat{\tau}$ and $\mathbb{P}^k$ denotes the polynomials of (x, y) of degree at most k. Let τ denote a general triangle in $\mathbb{R}^2$. Show that

$$\|u\|^2_{H^1(\tau)} \leq C_\theta \left\{\int_\tau |\nabla u|^2 \, dx + h^2 \bar{u}^2\right\}, \quad \text{for all } u \in \mathbb{P}^1$$

Here θ denotes the minimum angle of τ and h its diameter. Now $\bar{u}$ denotes the mean value of u on τ. (You may assume, without proof, standard properties involving the dependence on θ of the affine map of $\hat{\tau}$ onto τ.)

Solution: Doing the standard transformation onto the reference element, we have the following

$$\begin{aligned}
\|u\|^2_{H^1(\tau)} &= \int_\tau |u|^2 \, dx + \int_\tau |\nabla u|^2 \, dx \\
&\leq C_\theta\big(\int_{\hat{\tau}} |\hat{u}|^2 \, h^2 d\hat{x} + \int_{\hat{\tau}} \frac{1}{h^2} |\hat{\nabla}\hat{u}|^2 \, h^2 d\hat{x}\big) \\
&= C_\theta(h^2 \|\hat{u}\|^2_{L^2(\hat{\tau})} + |\hat{u}|^2_{H^1(\hat{\tau})})
\end{aligned}$$

Now using an add and subtract trick and using our hypothesis, we now have

$$\begin{aligned}
\|u\|^2_{H^1(\tau)} &\le C_\theta(h^2\|\hat{u}\|^2_{L^2(\hat{\tau})} + h^2|\hat{u}|^2_{H^1(\hat{\tau})} - h^2|\hat{u}|^2_{H^1(\hat{\tau})} + |\hat{u}|^2_{H^1(\hat{\tau})}) \\
&= C_\theta(h^2\|\hat{u}\|^2_{H^1(\hat{\tau})} + (1-h^2)|\hat{u}|^2_{H^1(\hat{\tau})}) \\
&\le C_\theta(h^2\int_{\hat{\tau}}|\hat{\nabla}\hat{u}|^2\,d\hat{x} + h^2\hat{\bar{u}}^2) + C_\theta(1-h^2)|\hat{u}|^2_{H^1(\hat{\tau})} \\
&\le C_\theta(h^2|u|^2_{H^1(\tau)} + h^2\bar{u}^2) + C_\theta(1-h^2)|u|^2_{H^1(\tau)} \\
&= C_\theta(|u|^2_{H^1(\tau)} + h^2\bar{u}^2)
\end{aligned}$$

This completes the proof. ■

b. Let V_h be the space of continuous piecewise linear functions with respect to a quasi-uniform mesh $\Omega = \bigcup_{i=1}^N \tau_i$. Consider the one point quadrature approximation

$$\mathcal{Q}_{\tau_i}(g) := |\tau_i|g(b_i) \approx \int_{\tau_i} g,$$

where $|\tau_i|$ is the area of τ_i and b_i is its barycenter.

Consider the finite element problem: Find $u_h \in V_h$ satisfying

$$A_h(u_h, \phi) = F_h(\phi), \quad \text{for all } \phi \in V_h.$$

Here for $u_h, v_h \in V_h$, A_h and F_h are given by

$$A(u_h, v_h) := \sum_{i=1}^N (\mathcal{Q}_{\tau_i}(\nabla u_h \cdot \nabla v_h) + \mathcal{Q}_{\tau_i}(u_h v_h)) \text{ and } F_h(v_h) := \sum_{i=1}^N \mathcal{Q}_{\tau_i}(f v_h)$$

respectively. Show that

$$\mathcal{Q}_{\tau_i}(|\nabla u|^2) = \int_{\tau_i} |\nabla u|^2 \text{ and } \mathcal{Q}_{\tau_i}(|u|^2) = |\tau_i|\bar{u}^2, \quad \text{for all } u \in \mathbb{P}^1.$$

Solution: Since u is a linear polynomial, ∇u is a constant function. Hence $\mathcal{Q}_{\tau_i}(|\nabla u|^2) = |\tau_i||\nabla u|^2(b_i) = \int_{\tau_i}|\nabla u|^2$. For $|u|^2$, since u is linear, express u in terms of the barycentric coordinates. I.e. say $u = a\lambda_1 + b\lambda_2 + c\lambda_3$. Note that the center of a triangle in barycentric coordinates is $(1/3, 1/3, 1/3)$. Then we have

$$\mathcal{Q}_{\tau_i}(|u|^2) = |\tau_i||u(b_i)|^2 = |\tau_i|\left|\frac{a}{3} + \frac{b}{3} + \frac{c}{3}\right|^2 = |\tau_i|\left|\frac{a+b+c}{3}\right|^2$$

Then note the identity, $\int_{\tau_i}\lambda_j\,dx = |\tau_i|/3$ for $j = 1,2,3$. So now look at the average $\bar{u}^2$,

$$|\tau_i|\bar{u}^2 = |\tau_i|\left(\frac{1}{|\tau_i|}\int_{\tau_i} a\lambda_1 + b\lambda_2 + c\lambda_3\,dx\right)^2 = |\tau_i|\left(\frac{1}{|\tau_i|}\cdot\frac{a+b+c}{3}|\tau_i|\right)^2$$

This gives us the result.■

c. Use parts (a) and (b) above to show that the form $A_h(\cdot,\cdot)$ is V_h-elliptic, i.e.,

$$A_h(v_h, v_h) \geq c\|v_h\|^2_{H^1(\Omega)}, \quad \text{for all } v_h \in V_h \tag{1}$$

holds with c independent of h.

Solution: From part b. we can write,

$$\begin{aligned} A_h(v_h, v_h) &= \sum_{i=1}^{N} \big(\mathcal{Q}_{\tau_i}(|\nabla v_h|^2) + \mathcal{Q}_{\tau_i}(|v_h|^2) \big) \\ &= \sum_{i=1}^{N} \Big(\int_{\tau_i} |\nabla v_h|^2 \, dx + |\tau_i||\bar{u}|^2 \Big) \\ &\geq C \sum_{i=1}^{N} \Big(\int_{\tau_i} |\nabla v_h|^2 \, dx + h^2|\bar{u}|^2 \Big). \end{aligned}$$

Thus from part a. we have

$$A_h(v_h, v_h) \geq C \sum_{i=1}^{N} \|v_h\|^2_{H^1(\tau_i)} = C\|v_h\|^2_{H^1(\Omega)}.$$

■

Problem 2. Let Ω be a convex polygonal domain of $\mathbb{R}^2$. Given $f \in L^2(\Omega)$, we denote by $u \in H^1_0(\Omega)$ the solution of the Poisson problem:

$$-\Delta u = f \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{2}$$

We note that u satisfies full elliptic regularity, i.e., $u \in H^2(\Omega)$.

We consider a *non conforming* finite element method to approximate u. Let $\{\mathcal{T}_h\}_{0<h<1}$ be a sequence of conforming shape regular subdivisions of Ω such that $\operatorname{diam}(T) \leq h$. Denote by X_h the spaces of continuous, piecewise linear polynomials subordinate to the subdivisions $\mathcal{T}_h$, $0 < h < 1$.

The numerical method consists of finding $u_h \in X_h$ such that for all $v_h \in X_h$:

$$a_h(u_h, v_h) := \int_{\Omega} \nabla u_h \cdot \nabla v_h - \int_{\partial\Omega} \partial_\nu u_h v_h + \frac{\alpha}{h} \int_{\partial\Omega} u_h v_h = \int_{\Omega} f v_h.$$

Here ν denotes the outward pointing unit normal (defined almost everywhere), $\partial_\nu u := \nabla u \cdot \nu$ and $\alpha > 0$ is a constant yet to be determined. Note that $X_h \not\subset H_0^1(\Omega)$ but $X_h \subset H^1(\Omega)$

a. Explain why $a_h(u, v_h)$ makes sense for any $v_h \in X_h$ and show Galerkin orthogonality, i.e.,

$$a_h(u - u_h, v_h) = 0, \quad \text{for all } v_h \in X_h. \tag{3}$$

Proof: Since $u \in H^2(\Omega)$, ∇u on Ω is well defined, u is also in $H_0^1(\Omega)$ so the integral term of u on $\partial\Omega$ will vanish, which is fine. Lastly, we might have an issue with $\partial_\nu u$ on $\partial\Omega$, but since $u \in H^2$, we know the trace exists for the ∇u. I.e. $\|\nabla u\|_{L^2(\partial\Omega)} \leq C\|\nabla u\|_{H^1(\Omega)} \leq C\|u\|_{H^2(\Omega)}$. Thus $a_h(u, v_h)$ makes sense for every $v_h \in X_h$.

Now onto Galerkin orthogonality. It is enough to show that u is also a solution to the finite element problem $a_h(\cdot, v_h) = f(v_h)$. So consider

$$\begin{aligned} a_h(u, v_h) &= \int_\Omega \nabla u \cdot \nabla v_h - \int_{\partial\Omega} \partial_\nu u v_h + \frac{\alpha}{h} \int_{\partial\Omega} u v_h \\ &= -\int_\Omega \Delta u v_h + \int_{\partial\Omega} \partial_\nu u v_h - \int_{\partial\Omega} \partial_\nu u v_h + 0 \\ &= -\int_\Omega \Delta u v_h \\ &= \int_\Omega f v_h \end{aligned}$$

Thus we have Galerkin orthogonality. ■

b. For any $v_h \in X_h$, define the mesh dependent norm

$$\|v_h\|_h := \left(\|\nabla v_h\|_{L^2(\Omega)}^2 + \frac{\alpha}{h} \|v_h\|_{L^2(\partial\Omega)}^2 \right)^{1/2}. \tag{4}$$

Show that there exists a constant c_0 independent of h such that for all $v_h \in X_h$

$$\int_{\partial\Omega} |\nabla v_h|^2 \leq \frac{c_0}{h} \int_\Omega |\nabla v_h|^2. \tag{5}$$

Using this fact, deduce that for all $v_h \in X_h$,

$$a_h(v_h, v_h) \geq \frac{1}{2} \|v_h\|_h^2, \tag{6}$$

provided $\alpha \geq c_0$.

Solution: Let e be an edge on $\partial\Omega$, which corresponds to an edge of a triangle in $\mathcal{T}_h$. Then consider the following

$$\begin{aligned}\int_{\partial\Omega} |\nabla v_h|^2 &= \sum_{e\subset\partial\Omega} \int_e |\nabla v_h|^2 \\ &\leq c \sum_{e\subset\partial\Omega} \int_0^1 |\frac{1}{h}\hat{\nabla}\hat{v}_h|^2 \, h d\hat{x} \\ &\leq \frac{c}{h} \sum_{e\subset\partial\Omega} \|\hat{\nabla}\hat{v}_h\|^2_{L^2(0,1)} \\ &\leq \frac{c}{h} \sum_{e\subset\partial\Omega} \|\hat{\nabla}\hat{v}_h\|^2_{L^2(\partial\hat{T})}.\end{aligned}$$

Now, notice that $\hat{\nabla}\hat{v}_h$ is constant on $\hat{T}$ so, we can apply the trace inequality. This gives us,

$$\begin{aligned}\int_{\partial\Omega} |\nabla v_h|^2 \, dx &\leq \frac{c}{h} \sum_{e\subset\partial\Omega} \|\hat{\nabla}\hat{v}_h\|^2_{H^1(\hat{T})} \\ &= \frac{c}{h} \sum_{e\subset\partial\Omega} \|\hat{\nabla}\hat{v}_h\|^2_{L^2(\hat{T})} \\ &\leq \frac{c}{h} \sum_{T\in\mathcal{T}_h} \|\nabla v_h\|^2_{L^2(T)} \\ &= \frac{c}{h} \int_\Omega |\nabla v_h|^2 \, dx\end{aligned}$$

For the coercivity, we start by applying Cauchy-Schwartz in the reverse direction, we have

$$\begin{aligned}a_h(v_h, v_h) &= \|\nabla v_h\|^2_{L^2(\Omega)} - \int_{\partial\Omega} \partial_\nu v_h v_h + \frac{\alpha}{h}\|v_h\|^2_{L^2(\partial\Omega)} \\ &\geq \|\nabla v_h\|^2_{L^2(\Omega)} - \|\nabla v_h\|_{L^2(\partial\Omega)}\|v_h\|_{L^2(\partial\Omega)} + \frac{\alpha}{h}\|v_h\|^2_{L^2(\partial\Omega)} \\ &\geq \|\nabla v_h\|^2_{L^2(\Omega)} - \sqrt{\frac{c_0}{h}}\|\nabla v_h\|_{L^2(\Omega)}\|v_h\|_{L^2(\partial\Omega)} + \frac{\alpha}{h}\|v_h\|^2_{L^2(\partial\Omega)} \\ &\geq \|\nabla v_h\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla v_h\|^2_{L^2(\Omega)} - \frac{\alpha}{2h}\|v_h\|^2_{L^2(\partial\Omega)} + \frac{\alpha}{h}\|v_h\|^2_{L^2(\partial\Omega)} \\ &= \frac{1}{2}(\|\nabla v_h\|^2_{L^2(\Omega)} + \frac{\alpha}{h}\|v_h\|^2_{L^2(\partial\Omega)}) \\ &= \frac{1}{2}\|v_h\|^2_h.\end{aligned}$$

This completes the proof. ■

c. Let I_h denote the Lagrange finite element interpolation operator associated with X_h. You may use the following estimate without proof: For $i = 1, 2$,

$$\left\|\frac{\partial(u - I_h u)}{\partial x_i}\right\|_{L^2(e)} \leq Ch^{1/2}\|u\|_{H^2(\tau)}. \tag{7}$$

Take $\alpha = c_0$ and derive an optimal error estimate for $\|u - u_h\|_h$.

Solution: We will first start by showing that a_h is continuous with respect to the norm $\|\cdot\|_h$. So consider,

$$\begin{aligned}
a_h(v_h, z_h) &= \int_\Omega \nabla v_h \cdot \nabla z_h - \int_{\partial\Omega} \frac{\partial v_h}{\partial n} z_h \, ds + \frac{\alpha}{h} \int_{\partial\Omega} v_h z_h \, ds \\
&\le |v_h|_{H^1(\Omega)} |z_h|_{H^1(\Omega)} + \|\nabla v_h\|_{L^2(\partial\Omega)} \|z_h\|_{L^2(\partial\Omega)} + \frac{\alpha}{h} \|v_h\|_{L^2(\partial\Omega)} \|z_h\|_{L^2(\partial\Omega)} \\
&\le |v_h|_{H^1(\Omega)} |z_h|_{H^1(\Omega)} + \sqrt{\frac{c_0}{h}} \|\nabla v_h\|_{L^2(\Omega)} \|z_h\|_{L^2(\partial\Omega)} + \frac{\alpha}{h} \|v_h\|_{L^2(\partial\Omega)} \|z_h\|_{L^2(\partial\Omega)} \\
&\le \|v_h\|_h |z_h|_{H^1(\Omega)} + \left(\|\nabla v_h\|_{L^2(\Omega)} + \sqrt{\frac{\alpha}{h}} \|v_h\|_{L^2(\partial\Omega)}\right) \sqrt{\frac{\alpha}{h}} \|z_h\|_{L^2(\partial\Omega)} \\
&\le \|v_h\|_h |z_h|_{H^1(\Omega)} + \sqrt{2} \|v_h\|_h \sqrt{\frac{\alpha}{h}} \|z_h\|_{L^2(\Omega)} \\
&\le 2\|v_h\|_h \|z_h\|_h.
\end{aligned}$$

Note, we have used the inequality $a + b \le \sqrt{2}\sqrt{a^2 + b^2}$ in the above argument. Thus a_h is continuous with respect to the norm $\|\cdot\|_h$.

Now we will derive Strang's first lemma for our problem. Using the fact that $a_h(u_h, v_h) = L(v_h)$ for all $v_h \in X_h$ where $L(v_h) := \int_\Omega f v_h$, we can write,

$$a_h(u_h - z_h, v_h) = a(u - z_h, v_h) + L(v_h) - a(u, v_h).$$

Then if we substitute $v_h = u_h - z_h$, we can apply coercivity and continuity to write,

$$\begin{aligned}
\frac{1}{2}\|u_h - z_h\|_h^2 &\le a_h(u_h - z_h, u_h - z_h) \\
&\le 2\|u - z_h\|_h \|u_h - z_h\|_h + \|L(\cdot) - a(u, \cdot)\|_{*,h} \|u_h - z_h\|_h.
\end{aligned}$$

We then divide by $\|u_h - z_h\|_h$ and rewrite the inequality as,

$$\|u_h - z_h\|_h \le 4\|u - z_h\|_h + 2\|L(\cdot) - a(u, \cdot)\|_{*,h}.$$

But from part a. we know that $a(u, v_h) = L(v_h)$ for every $v_h \in X_h$, hence $\|L(\cdot) - a(u, \cdot)\|_{*,h} = 0$. Therefore, our inequality becomes,

$$\|u_h - z_h\|_h \le 4\|u - z_h\|_h.$$

Using this inequality, we can estimate our error as,

$$\|u - u_h\|_h \le \|u - z_h\|_h + \|u_h - z_h\|_h \le 5\|u - z_h\|_h.$$

Since this holds for any $z_h \in X_h$, we can take the infimum, and our inequality becomes,

$$\|u - u_h\|_h \le 5 \inf_{z_h \in X_h} \|u - z_h\|_h.$$

Now we can proceed using the standard methods of error estimation by using the interpolation of our solution u, i.e., we have,

$$\|u - u_h\|_h \le 5 \inf_{z_h \in X_h} \|u - z_h\|_h \le 5\|u - I_h u\|_h.$$

Now consider,

$$\begin{aligned}
\|u - I_h u\|_h^2 &= \|\nabla(u - I_h u)\|_{L^2(\Omega)}^2 + \frac{c_0}{h}\|u - I_h u\|_{L^2(\partial\Omega)}^2 \\
&= \sum_{\tau \in \mathcal{T}_h} \|\nabla(u - I_h u)\|_{L^2(\tau)}^2 + \frac{c_0}{h} \sum_{e \subset \partial\Omega} \|u - I_h u\|_{L^2(e)}^2.
\end{aligned}$$

For $\|\nabla(u - I_h u)\|_{L^2(\tau)}^2$ we use the normal methods of transfering to the reference element and applying Bramble-Hilbert to get that, $\|\nabla(u - I_h u)\|_{L^2(\tau)}^2 \le Ch^2 |u|_{H^2(\tau)}^2$. So lets look at the boundary term,

$$\begin{aligned}
\frac{c_0}{h} \sum_{e \subset \partial\Omega} \|u - I_h u\|_{L^2(e)}^2 &\le \frac{c_0}{h} \sum_{e \subset \partial\Omega} Ch \|\hat{u} - \hat{I}_h \hat{u}\|_{L^2(\hat{e})}^2 \\
&\le C \sum_{e \subset \partial\Omega} \|\hat{u} - \hat{I}_h \hat{u}\|_{L^2(\hat{\tau})}^2 \\
&\le C \sum_{\tau \in \mathcal{T}_h} |\hat{u}|_{H^2(\hat{\tau})}^2 \\
&\le Ch^2 \sum_{\tau \in \mathcal{T}_h} |u|_{H^2(\tau)}^2 \\
&= Ch^2 |u|_{H^2(\Omega)}^2.
\end{aligned}$$

Therefore,

$$\|u - I_h u\|_h \le Ch|u|_{H^2(\Omega)}.$$

Thus,

$$\|u - u_h\|_h \le Ch|u|_{H^2(\Omega)}.$$

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Problem 3. Given the boundary value problem: find $u(x,t)$ such that

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \kappa \frac{\partial^2 u}{\partial x^2} - b(x)\frac{\partial u}{\partial x} + f(x),\ 0 < x < 1,\ 0 < t \le T, \\
u(0,t) &= 0,\ u(1,t) = 0,\ 0 < t \le T \\
u(x,0) &= v(x),\ 0 \le x \le 1,
\end{aligned}$$

where $\kappa = const > 0$, $b(x) \in C^0[0,1]$, $v(x)$, and $f(x)$ are given smooth functions Let $x_i = ih$ with $h = 1/N$ and $t_n = n\tau$, with $n = 0, 1, \ldots, J$ and (time step size) $\tau = T/J$.

a. Write down a forward (explicit) Euler fully discrete scheme for the above problem based on a finite difference discretization in space which upwinds the $b(x)$ term.

Solution: Let $U_j^n = u(x_j, t_n)$. Then our finite difference discretization is

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \kappa \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} - \left(b^+(x) \frac{U_j^n - U_{j-1}^n}{h} + b^-(x) \frac{U_{j+1}^n - U_j^n}{h} \right) + f(x) \tag{8}$$

where we have

$$b^+(x) = \begin{cases} b(x) & \text{if } b(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad b^-(x) = \begin{cases} b(x) & \text{if } b(x) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

■

b. Find a Courant (CFL) condition and show that if this condition is satisfied,

$$||U^{n+1}||_\infty \leq ||U^n||_\infty + \tau ||f(t_n)||_\infty.$$

Here U^n is the approximation at t_n of part a.

Solution: Let's rewrite our discretization in a. as follows,

$$U_j^{n+1} = \big(\frac{\tau\kappa}{h^2} - \frac{\tau b^-(x_j)}{h}\big) U_{j+1}^n + \big(1 - \frac{2\tau\kappa}{h^2} - \frac{\tau b^+(x_j)}{h} + \frac{\tau b^-(x_j)}{h}\big) U_j^n + \big(\frac{\tau\kappa}{h^2} + \frac{\tau b^+(x_j)}{h}\big) U_{j-1}^n + \tau f(x_j).$$

Notice the coefficients of U_{j+1}^n and U_{j-1}^n are both positive. So if we have,

$$\tau\big(\frac{2\kappa}{h^2} + \frac{b^+(x_j)}{h} - \frac{b^-(x_j)}{h}\big) \leq 1,$$

then,

$$|U_j^{n+1}| \leq \big(\frac{\tau\kappa}{h^2} - \frac{\tau b^-(x_j)}{h}\big) ||U^n||_\infty + \big(1 - \frac{2\tau\kappa}{h^2} - \frac{\tau b^+(x_j)}{h} + \frac{\tau b^-(x_j)}{h}\big) ||U^n||_\infty + \big(\frac{\tau\kappa}{h^2} + \frac{\tau b^+(x_j)}{h}\big) ||U^n||_\infty + \tau ||f||_\infty,$$

which simplifies to

$$||U^{n+1}||_\infty \leq ||U^n||_\infty + \tau ||f||_\infty.$$

Now note that $b^+(x_j) - b^-(x_j) = |b(x_j)|$. So we can derive our CFL condition, so we can write,

$$\tau\big(\frac{2\kappa}{h^2} + \frac{|b(x_j)|}{h}\big) \leq 1.$$

This can be rewritten as,

$$\tau \leq \frac{h^2}{2\kappa + h|b(x_j)|}.$$

Since this needs to hold for every x_j, our CFL condition must be,

$$\tau \leq \frac{h^2}{2\kappa + h||b||_\infty}.$$

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c. Define the fully discrete method but with backward (implicit) Euler time stepping and show that this scheme is unconditionally stable, i.e., prove that for any positive τ,

$$||U^{n+1}||_\infty \leq ||U^n||_\infty + \tau||f(t_{n+1})||_\infty. \tag{9}$$

Solution: The backward Euler is given by,

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \kappa\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2} - b^+(x_j)\frac{U_j^{n+1} - U_{j-1}^{n+1}}{h} - b^-(x_j)\frac{U_{j+1}^{n+1} - U_j^{n+1}}{h} + \tau f(x_j).$$

This can be rewritten as,

$$\Big(1 + \frac{2\kappa\tau}{h^2} + \frac{\tau|b_j|}{h}\Big)U_j^{n+1} - U_j^n = \Big(\frac{\kappa\tau}{h^2} - \frac{\tau b_j^-}{h}\Big)U_{j+1}^{n+1} + \Big(\frac{\kappa\tau}{h^2} + \frac{\tau b_j^+}{h}\Big)U_{j-1}^{n+1} + \tau f(x_j).$$

Now applying triangle inequality, we have,

$$\begin{aligned}
\Big(1 + \frac{2\kappa\tau}{h^2} + \frac{\tau|b_j|}{h}\Big)|U_j^{n+1}| &\leq |U_j^n| + \Big(\frac{\kappa\tau}{h^2} - \frac{\tau b_j^-}{h}\Big)|U_{j+1}^{n+1}| + \Big(\frac{\kappa\tau}{h^2} + \frac{\tau b_j^+}{h}\Big)|U_{j-1}^{n+1}| + \tau|f_j| \\
&\leq ||U^n||_\infty + \Big(\frac{\kappa\tau}{h^2} - \frac{\tau b_j^-}{h}\Big)||U^{n+1}||_\infty + \Big(\frac{\kappa\tau}{h^2} + \frac{\tau b_j^+}{h}\Big)||U^{n+1}||_\infty + \tau||f||_\infty \\
&= ||U^n||_\infty + \Big(\frac{2\kappa\tau}{h^2} + \frac{\tau|b_j|}{h}\Big)||U^{n+1}||_\infty + \tau||f||_\infty
\end{aligned}$$

Since this must hold for every j, we thus have,

$$||U^{n+1}||_\infty \leq ||U^n||_\infty + \tau||f||_\infty$$

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